Overview

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I. Introduction: What is Kinetic Theory?

- Kinetic theory deals with systems of many objects
e.g. gases, galaxies, cells, neurons, ...

→ description of behaviour of rarefied gases, plasmas,
cell migration, swarming behaviour, neuron networking, ...

- 3 different viewpoints:
  - microscopic
  - mesoscopic
  - macroscopic
Microscopic scale

Consider system as collection of particles, try to solve the associated equations of motion of all the particles.

Example: Hamilton equations from classical mechanics

\[ x = (x_1, ..., x_N) \in (\mathbb{R}^d)^N \text{ and momenta } p = (p_1, ..., p_N) \in (\mathbb{R}^d)^N \]

\[ \dot{x}(t) = \nabla_p H(x(t), p(t)), \quad \dot{p}(t) = -\nabla_x H(x(t), p(t)) \]

with Hamiltonian (energy) \( H \in \mathbb{C}^2(\mathbb{R}^{2dN}) \),

\[ H(x, p) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} V(x_i) + \sum_{i \neq j \in \mathbb{N}}^{N} \Phi(|x_i - x_j|) \]

\( \Phi \) kin. Energy \( \Phi \) external potential interaction potential

Problems: \( N \) very large (typically \( N \approx 10^{23} \))

Initial conditions not known with absolute accuracy

\( \rightarrow \) probability density \( \Phi \)

Describing distribution of probability for initial data

Tools: Dynamical Systems, Probability Theory and Stochastic Processes
mean-field limit: passing from many-particle model
to equation of an average particle

- mesoscopic scale

purely kinetic viewpoint, evolution of a single average
(typical) particle (→ statistical study)

Example: Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = \mathcal{Q}(f,f)$$

where $f = f(t, x, v)$ is phase-space distribution
function (more later), $F = -\nabla_v$ external forces,
and

$$\mathcal{Q}(f,f)(v) = \int \text{d}v' \int \text{d}k \; \mathcal{B}(1v - v_\star, \frac{v - v_\star}{1v - v_\star}, k) \times$$

$$\left[ f(v') f(v') - f(v_\star) f(v) \right]$$

Boltzmann collision operator,

where $v_\star = \frac{v + v'}{2}, \frac{v - v'}{2} < v$.

Problems: nonlinear and nonlocal PDE (Integro-Differential Equation)
derivation from Hamilton equations so far
not satisfactory

Tools: Analysis, nonlinear PDEs, Probability,
       Functional analysis,...
Macrosopic scale

considers a small part of the system as unit, small in comparison to the system, but large enough to contain a large amount of particles → statistical treatment

Example: 
\[ p(t, x) = \int f(t, x, v) \, dv \]  
local density

\[ p u(t, x) = \int f(t, x, v) \, v \, dv \]  
local macroscopic velocity

\[ \int u^2 + N p \, T = \int f(t, x, v) \, v^2 \, dv \]  
local temperature

Local conservation laws

\[ \partial_t p + \partial_x (p u) = 0 \]

\[ \partial_t (p u) + \partial_x \left( \int f \, v \, dv \right) = 0 \]

\[ \partial_t (p (u^2 + N p T)) + \partial_x \left( \int f \, v^2 \, dv \right) = 0 \]

yield compressible Euler equations in suitable scaling limit:

\[ \partial_t p + \partial_x (p u) = 0 \]

\[ \partial_t (p u) + \partial_x (p u u + p T u) = 0 \]

\[ \partial_t (p (u^2 + N p T)) + \partial_x (p (u^2 + (N+2) p T u)) = 0 \]
Consider \( N \) particles in \( \mathbb{R}^d \):
- positions \( (x_1, \ldots, x_N) \in \mathbb{R}^{dN} \)
- velocities \( (v_1, \ldots, v_N) \in \mathbb{R}^{dN} \)

Coordinates \( \bar{z} = (z_1, \ldots, z_N) \in \mathbb{R}^{2dN} \),
\( z_i = (x_i, v_i) \in \mathbb{R}^{2d} \)

Equations of motion: free flow (all particles have mass 1)

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0, \quad 1 \leq i \leq N,
\]
on the domain
\[ \mathcal{D}_N = \{ \bar{z} \in \mathbb{R}^{2dN} : |x_i - x_j| > \varepsilon \forall i \neq j \} \]
and elastic reflection on boundary \( \partial \mathcal{D}_N \):
if \( |x_i - x_j| = \varepsilon \), then
\[
\begin{align*}
v_i' &= v_i - \omega_{ij} \cdot (v_i - v_j) \\
v_j' &= v_j + \omega_{ij} \cdot (v_i - v_j)
\end{align*}
\]
where \( \omega_{ij} = \frac{x_i - x_j}{|x_i - x_j|} \) and is the case when
the ingoing velocities \( v_i, v_j \) are pre-collisional, i.e.,
\[ \omega_{ij} \cdot (v_i - v_j) < 0. \]

Notice that the reflection is elastic, that is,
\[ v_i' + v_j' = v_i + v_j \] conservation of momentum
\[ \frac{1}{2} |v_i'|^2 + \frac{1}{2} |v_j'|^2 = \frac{1}{2} |v_i|^2 + \frac{1}{2} |v_j|^2 \] conservation of kinetic energy
The pre-collisional velocities \((v_i', v_j')\) and the post-collisional velocities \((v_i, v_j)\) are related by a linear transformation
\[
(v_i', v_j') = \mathbf{R} \cdot (v_i, v_j),
\]
where \(\mathbf{R} = \begin{pmatrix}
1 - \omega_i \cdot \omega_j & \omega_i \cdot \omega_j \\
\omega_j \cdot \omega_i & 1 - \omega_i \cdot \omega_j
\end{pmatrix}
\]

Obvious (and non-trivial) question: do these equations of motion (with boundary conditions) define a global dynamics on \(\mathbb{R}^2\)?

\[\rightarrow\] not simple consequence of Cauchy-Lipschitz (Picard-Lindeloef) theorem, since boundary conditions non-smooth (not even defined for all particle configurations)

Possible problems: Pathological trajectories

- there exists a collision involving more than two particles
- the collision is grazing (\(\omega_i \cdot (v_i' - v_j) \to 0\)), hence boundary condition is not well-defined
- infinitely many collisions in finite time, so the dynamics cannot be defined globally.
Proposition 1. Let \( N \in \mathbb{N} \) and \( \varepsilon > 0 \) be fixed. The set of initial configurations leading to a pathological trajectory is of measure zero in \( \mathbb{R}^{2N} \).

We first prove the following.

Lemma 2. Let \( p, R > 0 \) be given, and \( 5 \leq \varepsilon / 2 \).

Define the set

\[
I := \{ \xi \in b_\varepsilon \times B_R : \text{one particle will collide with two others on the time interval } [0, \delta] \}.
\]

Then \( |I| \leq C(N, \varepsilon, R) \int_{\mathbb{R}^{2(N-2)}} \delta^2 \).

We use the notation \( B_R := \{ \xi \in \mathbb{R}^N : \|\xi\| \leq R \} \),

\[
\{x_1, \ldots, x_N\} = \{x_1 \in \mathbb{R}^N, \|x_i\| \leq R, i = 1, \ldots, N \},
\]

and \( |I| \) is the Lebesgue measure of \( I \).

Proof: Obviously,

\[
I \subseteq \bigcup_{j=1}^N \bigcup_{i,j} \{ x_j \in b_\varepsilon \times B_R : \exists \{x_1, \ldots, x_N\} \text{ distinct s.t.} \}
\]

\[
\varepsilon \leq |x_i - x_j| \leq \varepsilon + 2R \delta \quad \text{and} \quad \varepsilon \leq |x_i - x_k| \leq \varepsilon + 2R \delta,
\]

\[
\{ x_j \in b_\varepsilon \times B_R : \exists \{x_1, \ldots, x_N\} \text{ distinct s.t.} \}
\]

since \( |x_1| \leq R \), so if \( |x_i - x_j| \leq \varepsilon + 2R \delta \), there always exist velocities \( v_i, v_j \) s.t. \( |x_1| \leq R \) and a collision occurs.
So we can estimate

\[ \| I \| \leq \| I \| = 18 a_1 \left( \frac{N}{3} \right) \int_{k=1}^{N-3} \int_{R^3} d \mathbf{x}_1 \, d \mathbf{x}_2 \, d \mathbf{x}_3 \]

\[ \int_{k=1}^{N-3} \int_{R^3} \mathbf{x}_1 \cdot \mathbf{b}_{\mathbf{x}_1} \mathbf{x}_2 \cdot \mathbf{b}_{\mathbf{x}_2} \mathbf{x}_3 \cdot \mathbf{b}_{\mathbf{x}_3} \]

\[ \int_{k=1}^{N-3} \int_{R^3} \mathbf{x}_1 \cdot \mathbf{b}_{\mathbf{x}_1} \mathbf{x}_2 \cdot \mathbf{b}_{\mathbf{x}_2} \mathbf{x}_3 \cdot \mathbf{b}_{\mathbf{x}_3} \]

\[ \int_{k=1}^{N-3} \int_{R^3} \mathbf{x}_1 \cdot \mathbf{b}_{\mathbf{x}_1} \mathbf{x}_2 \cdot \mathbf{b}_{\mathbf{x}_2} \mathbf{x}_3 \cdot \mathbf{b}_{\mathbf{x}_3} \]

\[ \int_{k=1}^{N-3} \int_{R^3} \mathbf{x}_1 \cdot \mathbf{b}_{\mathbf{x}_1} \mathbf{x}_2 \cdot \mathbf{b}_{\mathbf{x}_2} \mathbf{x}_3 \cdot \mathbf{b}_{\mathbf{x}_3} \]

\[ \leq 18 a_1 \left( \frac{N}{3} \right) \int_{R^3} \mathbf{x}_1 \cdot \mathbf{b}_{\mathbf{x}_1} \mathbf{x}_2 \cdot \mathbf{b}_{\mathbf{x}_2} \mathbf{x}_3 \cdot \mathbf{b}_{\mathbf{x}_3} \]

\[ \leq C(N, R, \varepsilon, d) \int_{R^3} \mathbf{x}_1 \cdot \mathbf{b}_{\mathbf{x}_1} \mathbf{x}_2 \cdot \mathbf{b}_{\mathbf{x}_2} \mathbf{x}_3 \cdot \mathbf{b}_{\mathbf{x}_3} \]

\[ = C(N, R, \varepsilon, d) \int_{R^3} \frac{1+2R^2}{d^2} \left( \int_{1}^{\infty} r^{-1} dr \right)^2 \]

\[ = C(N, R, \varepsilon, d) \int_{R^3} \left( (1+2R^2) d^2 - 1 \right)^2 \]

Now use that \((1+x)^d - 1 \leq \frac{C(1+R)^d - 1}{R} x^d\) for all \(x \in [0, R]\), together with \(\delta < \frac{1}{2}\), to obtain

\[ \| I \| \leq C(N, R, \varepsilon, d) \int_{R^3} \delta^2. \]
Proof of Proposition 1.

Let $R > 0$ and fix some time $t > 0$. Let $S < S_2$ be s.t. $t/S$ is an integer.

By Lemma 2 there exists a subset $I_0(S, R)$ of $b^n \times b^R$ with $|I_0(S, R)| \leq C(N, S, R) R^{d(N-2)} S^2$ s.t. any initial configuration in $(b^n \times b^R) \setminus I_0(S, R)$ generates a solution on $[0, S]$, with the property that each particle collides with at most one other particle on $[0, S]$.

Since $\forall t \in \mathbb{R}^{2dN} : |x_i - x_j| = S$, $w_i \cdot (v_i - v_j) = 0$

for some $i \neq j$ if $|x_i - x_j| = S$,

up to removing a set of measure zero of initial configurations, every collision is non-regular.

Notice that here we used the fact that the flow is measure-preserving: let $S_t \Sigma := \Sigma(t)$, then for any $A \subseteq \mathbb{R}^n : |S_t^{-1} A| = |A|$. Indeed, for any measurable $i \in \{1, \ldots, N\}$, the map $\tilde{x}_i : (x_i, v_i) \mapsto (x_i(t), v_i(t)) = \tilde{x}_i(t)$, $t \in [0, S]$, is made up of a finite sequence of translations $(x_i, v_i) \mapsto (x_i + tv_i, v_i)$ and reflections $(x_i, v_i) \mapsto (x_i, v_i')$, which depend smoothly on $(x_i, v_i)$ and have Jacobian 1.
Now start again at time $t$.

Notice that the ball of radius $R$ in the velocity variables is stable under the flow.

The positions at time $t$ lie in the set $b^N_{R+5R}$. Now apply Lemma 2 to the new configuration space at time $t$. Since the measure is invariant under the flow, we can construct a subset $I_t(S,R)$ of the initial positions and velocities $b^N_{R} \times b_{R}$ with

$$|I_t(S,R)| \leq C(N,\varepsilon,R) R^{d(N-2)} (1+\varepsilon)^{d(N-2)} R^2,$$

s.t. outside $I_0(S,R) \cup I_t(S,R)$, the flow starting from any initial configuration in $b^N_{R} \times b_{R}$ is such that each particle collides with at most one other particle on $[0,5]$ and then at most one other particle on $[5,25]$ (again in a non-grazing collision).

Repeat the procedure $t/5$ times:

construct a subset $I_{t_5}(S,R) := \bigcup_{j=0}^{t_5} I_j(S,R) \subseteq b^N_{R} \times b_{R}$ of measure

$$|I_{t_5}(S,R)| \leq C(N,\varepsilon,R) R^{d(N-2)} R^2 \sum_{j=0}^{t_5-1} (1+\varepsilon)^{d(N-2)} \leq \frac{t_5}{t/5} (1+t).$$
such that for any initial condition in \((b^{n}_{R} x b^{n}_{R}) \setminus I_{5}(t,R)\)
the flow is well-defined up to time \(t\).

The intersection \(I(t,R) := \bigcap_{\delta > 0} I_{3}(t,R)\) has measure

\[|I(t,R)| = \inf_{\delta > 0} |I_{3}(t,R)| = 0,\]

so any initial configuration in \((b^{n}_{R} x b^{n}_{R}) \setminus I(t,R)\)
generates a well-defined flow until time \(t\).

Finally, take sequences \(t_{n} \uparrow \infty\), \(R_{n} \uparrow \infty\) and set

\[I := \bigcap_{n} I(t_{n}, R_{n}).\]

Then \(|I| = 0\) and any initial configuration in \(\mathbb{R}^{n}\) outside \(I\) generates
a well-defined flow.
Liouville Equation and BBGKY Hierarchy

Limit of large particle numbers $N$: individual trajectories become irrelevant.

Goal: describe average behaviour.

Since particles are indistinguishable, we will be interested in some distribution related to the empirical measure

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t), v_i(t)} ,$$

where $x_N = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$, $v_N = (v_1, \ldots, v_N) \in \mathbb{R}^{dN}$ and $(x_i(t), v_i(t))$ is the state of particle $i$ at time $t$ in the system with initial configuration $(x_N, v_N)$.

Problem: only imprecise knowledge of the state of the system at initial time.

$\Rightarrow$ average over initial configurations with distribution $f_0^0(z_N)$.

Notation: for $s \not\equiv 1, \ldots, N$ write $x_s = (x_1, \ldots, x_s) \in \mathbb{R}^{ds}$, $v_s = (v_1, \ldots, v_s) \in \mathbb{R}^{ds}$ and $z_s = (z_1, \ldots, z_s) \in \mathbb{R}^{2ds}$, $z_i = (x_i, v_i) \in \mathbb{R}^{2d}$.

$\Rightarrow$ want to describe evolution of the distribution

$$\int \left( \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i(t)} \right) f_0^0(z_N) \, d\mathbb{Z}_N.$$
This yields the \( N \)-particle distribution function
\[
f_N = f_N(t, z_N)
\]
(in probability language: \( f_N(t, z_N) = \text{Law}(z_1, t, \ldots, z_N(t)) \)).

By indistinguishability of particle \( f_N \) satisfies
\[
f_N(t, z_{\sigma(w)}) = f_N(t, z_N)
\]
for all permutations \( \sigma \) of \( 1, \ldots, N \)
\[
(z_{\sigma(w)} = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}, v_{\sigma(w)}))
\]

**Proposition 3**

Let \( \Gamma_N \) be the set of full measure in \( \mathbb{R}^N \) where the hard sphere dynamics is well-defined for all \( t > 0 \). Assume that \( f_N \in C^1(\mathbb{R}_t \times \Gamma_N) \). Then \( f_N \) satisfies the Liouville equation
\[
\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_x f_N = 0 \quad \text{on} \quad \Gamma_N
\]
with boundary condition \( f_N(t, z_N') = f_N(t, z_N) \), i.e.
\[
f_N(t, \ldots, x_i, v_i', \ldots, x_j, v_j', \ldots) = f_N(t, \ldots, x_i, v_i, \ldots, x_j, v_j, \ldots)
\]
on \( \frac{1}{2} |x_i - x_j|^2 < 0 \).
Proof: Let \( dp_n(0) = f_n(0, z_N) \, d\bar{z}_N \) be the initial probability distribution; and assume that the pushforward of \( p_n(0) \) under \( \phi_t \), \( \phi_t \# p_n(0) \), is a.c. w.r.t. Lebesgue measure on \( \Gamma \) with density \( f_n(t, z_N) \). (\( \phi_t \) is the solid sphere flow.)

Then \( f_n(t, z_N) = f_n(0, \phi_t^{-1}(z_N)) \):

Indeed, let \( A \in \mathcal{B}_n \), then

\[
(\phi_t \# p_n(0))(A) = \int 1_A(z_N) \, d(\phi_t \# p_n(0))(z_N)
\]

\[
= \int 1_A(\phi_t(z_N)) \, dp_n(0)(z_N)
\]

\[
= \int 1_A(z_N) \, f_n(0, \phi_t(z_N)) \, d\bar{z}_N
\]

\[
= \int 1_A(z_N) \, f_n(0, \phi_t^{-1}(z_N)) \, d\bar{z}_N
\]

Comparing with \( (\phi_t \# p_n(0))(A) = \int 1_A(z_N) \, f_n(t, z_N) \, d\bar{z}_N \)

yields the result.

It follows that

\[
\partial_t f_n(t, z_N) = \frac{1}{dt} f_n(0, \phi_t^{-1}(z_N)) = - \nabla f_n(0, \phi_t^{-1}(z_N)) \cdot \phi_t^{-1}(z_N)
\]

\[
= - \sum_{i=1}^{2^n} v_i \cdot \nabla_i f_n(0, \phi_t^{-1}(z_N))
\]

\[
= - \sum_{i=1}^{2^n} v_i \cdot \nabla_i f_n(t, z_N)
\]

So

\[
\partial_t f_n(t, z_N) + \sum_{i=1}^{2^n} v_i \cdot \nabla_i f_n(t, z_N) = 0.
\]
We are interested in the evolution of a "typical particle", that is, the evolution of the one-particle marginal distribution

\[ f^{(1)}_N(t, z_1) = \int f_N(t, z_1, \ldots, z_N) \, d\mathbf{z}_N \, d\mathbf{r}_N \, dz_2 \ldots dz_N. \]

More generally, we define

**Definition 4 (Marginal distribution)**

Let \( s \in \{1, \ldots, N-1\} \). Then

\[ f^{(s)}_N(t, z_s) = \int f_N(t, z_s, z_{s+1}, \ldots, z_N) \, d\mathbf{z}_{s+1} \ldots d\mathbf{z}_N. \]

Notice that \( f^{(s)}_N \) is defined on \( \mathbb{R}^s \) only and that

\[ f^{(s)}_N(t, z_s) = \int f^{(s+1)}_N(t, z_s, z_{s+1}) \, dz_{s+1}. \]

By integration of boundary condition on \( f_N \),

\[ f_N(t, z'_N) = f_N(t, z_N) \]

whenever \( z_N \in \mathbb{R}^N \), we obtain the b.c.

\[ f^{(s)}_N(t, z'_s) = f^{(s)}_N(t, z_s). \]

From Liouville's equation we can derive a hierarchy of equations for the marginals of \( f^{(N)}_N \), the so-called BBGKY hierarchy (after Bogoliubov - Born - Green - Kirkwood - Yvon).
Theorem 5 (BBGKY hierarchy)

Assume that \( f_n \in C^1(\mathbb{R}^+ \times \mathbb{R}^N) \) decays at infinity in \( v \) and satisfies the Liouville equation.

Then its marginals \( f^{(s)} \) satisfy the weak formulation of the BBGKY hierarchy

\[
\mathcal{D}_t f^{(s)} + \sum_{i=1}^S v_i \cdot \nabla_i f^{(s)} = C_{S, S+1} f^{(S+1)} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}_S
\]

with collision operator

\[
(C_{S, S+1} f^{(S+1)})(t, z_S) = (N-S)^{d-1} \sum_{i=1}^S \int_{\mathbb{R}^{d-1} \times \mathbb{R}^d} \omega \cdot (v_{S+1} - v_i) f^{(S+1)}(t, z_S, x_i + \epsilon v_i, v_{S+1}) \, dv_i \, dv_{S+1}.
\]

Remarks: 1) Weak formulation of the BBGKY hierarchy:

Let \( \psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}_S) \) be symmetric and periodic and satisfy the boundary condition \( \psi(t, z_S) = \psi(t, z_S) \).

\[
(\omega - \text{BBGKY}) \quad - \int_{\mathbb{R}^+ \times \mathbb{R}^2} f^{(S)}(t, z_S) \partial_z \psi(t, z_S) \, dt \, dz - \int_{\mathbb{R}^+ \times \mathbb{R}^2} f^{(S)}(0, z_S) \psi(0, z_S) \, dz
\]

\[
- \sum_{i=1}^S \int_{\mathbb{R}^+ \times \mathbb{R}^2} f^{(S)}(t, z_S) v_i \cdot \nabla_i \psi(t, z_S) \, dt \, dz
\]

\[
= \int_{\mathbb{R}^+ \times \mathbb{R}^2} (C_{S, S+1} f^{(S+1)})(t, z_S) \psi(t, z_S) \, dt \, dz
\]
2. We can split the collision operator into two terms, depending on the sign of $\omega \cdot (v_{s+1} - v_i)$:

- $\omega \cdot (v_{s+1} - v_i) < 0 \implies (x_{s+1} - x_i) \cdot (v_{s+1} - v_i) < 0$; distance between $x_{s+1}$ and $x_i$ decreases up to collision time → pre-collisional

- $\omega \cdot (v_{s+1} - v_i) > 0$: post-collisional

\[
(C_{s,s+1} f^{(s+1)}_N) (t, z_s) = (N-S) \varepsilon^{d-1} \sum_{i=1}^S \left[ \int_{\mathbb{R}^d \setminus x_{s+1}} [\omega \cdot (v_{s+1} - v_i)]_+ f^{(s+1)}_N (t, z_s, x_i + \varepsilon \omega, v_{s+1}) \, d\omega d v_{s+1} \right. \\
- \left. \int_{\mathbb{R}^d \setminus x_{s+1}} [\omega \cdot (v_{s+1} - v_i)]_- f^{(s+1)}_N (t, z_s, x_i - \varepsilon \omega, v_{s+1}) \, d\omega d v_{s+1} \right]
\]

\[
= (N-S) \varepsilon^{d-1} \sum_{i=1}^S \int_{\mathbb{R}^d \setminus x_{s+1}} [\omega \cdot (v_{s+1} - v_i)]_+ \left( f^{(s+1)}_N (t, z_s', x_i + \varepsilon \omega, v_{s+1}') - f^{(s+1)}_N (t, z_s, x_i - \varepsilon \omega, v_{s+1}) \right) \, d\omega d v_{s+1}
\]

where we changed $\omega \rightarrow -\omega$ in the second (pre-collisional) term and made use of the boundary condition $f^{(s+1)}_N (t, z_s, x_i + \varepsilon \omega, v_{s+1}) = f^{(s+1)}_N (t, z_s', x_i + \varepsilon \omega, v_{s+1}')$ in the first term.
Boltzmann-Grad Limit

To get a mesoscopic evolution equation, we want to take the limit \( N \to \infty, \varepsilon \to 0 \), in such a way that the interchanges have a macroscopic effect on the dynamics: each particle should undergo a finite number of collisions per unit of time.

A particle with velocity \( \varepsilon \) covers an area of size \( \varepsilon^d \) in unit time. As there are \( N \) particles, want \( N \varepsilon^d = O(1) \) to see effect of collisions (Boltzmann-Grad scaling).

\[ \Rightarrow \text{Boltzmann hierarchy} \]

\[ \partial_t f^{(s)} + \sum_{i=1}^{s} \mathbf{v}_i \cdot \nabla_{x_i} f^{(s)} = C_{s,s+1} f^{(s+1)} \]

\[ = \sum_{i=1}^{s} \int_{\mathbb{R}^d \times \mathbb{R}^{d-1}} \left[ \omega \cdot (v_{s+1} - v_i) \right] f^{(s+1)} (t, x_1, v_1, \ldots, x_i, v_i, \ldots, x_s, v_s, x_i, v_{s+1}) - f^{(s+1)} (t, z_s, x_i, v_{s+1}) \, dv_{s+1} \, dz \]

In particular, if the second marginal factorises (particles are uncorrelated), then "molecular chaos", the Boltzmann hypothesis

\[ \partial_t f^{(1)} (t, x, v) + v \cdot \nabla_x f^{(1)} (t, x, v) = (C_{1,2} f^{(2)}) (t, x_i, v) \]

\[ = (C_{1,2} f^{(1)} \otimes f^{(1)}) (t, x, v) \]
This is the Boltzmann equation for hard spheres:

\[
\frac{\partial f(t,x,v)}{\partial t} + v \cdot \nabla f(t,x,v) = \int_{S^{d-1} \times \mathbb{R}^d} \left[ \omega \cdot (v_a - v) 
\right] 
\cdot f(t, x, v') (f(t, x, v') \cdot f(t, x, v_a) \\
- f(t, x, v) f(t, x, v_a)) \, d\omega \cdot dv_a
\]

Where \( v' = v - \omega \cdot (v - v_a) \omega \) and \( v_a' = v_a + \omega \cdot (v - v_a) \omega \), \( \omega \in S^{d-1} \),

s.t. \( v' + v_a' = v + v_a \) and \( |v'|^2 + |v_a'|^2 = |v|^2 + |v_a|^2 \)

(conservation of momentum and kinetic energy).