Mathematical Topics in Kinetic Theory

Exercise Sheet 1

Exercise 1 (Hamiltonian Flows)

Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n$ open and $H \in \mathscr{C}^2(\Gamma)$ with $\sup_{z\in\Gamma} \|D^2H(z)\| < \infty$. The system of differential equations

$$\dot{x} = \nabla_p H(x,p), \quad \dot{p} = -\nabla_x H(x,p)$$

is called Hamiltonian differential equation. $H : \Gamma \to \mathbb{R}$ is called the Hamiltonian.

(a) Show that the system is equivalent to

$$\dot{z} = \mathbb{J}\nabla H(z) \tag{1}$$

with $z = (x,p) \in \Gamma$ and the matrix $\mathbb{J} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$.

(b) Let $z : \mathbb{R} \to \Gamma$ be the solution of (1) with initial condition $z(0) = z_0$. Show that for any $t \in \mathbb{R}$ the maps

$$\phi_t : \Gamma \to \Gamma, \quad z_0 \mapsto \phi_t(z_0) = z(t)$$

are well-defined and that $\Phi = \{\phi_t : t \in \mathbb{R}\}$ is a one-parameter group, which is called the *Hamiltonian flow*.

Exercise 2 (Liouville's Theorem)

(a) Let Φ be a Hamiltonian flow with Hamiltonian H. Show that any function $F = f \circ H$ of H is invariant under Φ, that is,

$$F \circ \phi_t = F \quad \text{for all } t \in \mathbb{R}.$$

(b) Show that the Jacobian $|\det D\phi_t(z)|$ of a Hamiltonian flow is constant and equal to 1. (Liouville's theorem)
Hint: Use that for two linear mappings $M(t)$ and $A(t)$ such that $\dot{M}(t) = A(t)M(t)$ for all $t \geq 0$ one has

$$\det M(t) = \det M(0) \exp\left(\int_0^t \mathrm{tr} A(s)\,\mathrm{d}s\right).$$

(c) Denote by $\mathscr{P}(\Gamma, \mathscr{B}_\Gamma)$ the set of probability measures on Γ with Borel σ-algebra $\mathscr{B}_\Gamma$.

Let $\mu \in \mathscr{P}(\Gamma, \mathscr{B}_\Gamma)$ with density $\rho = f \circ H$ with respect to Lebesgue measure on Γ, that is, $\mu(A) = \int_A \rho\,\mathrm{d}z$ for all $A \in \mathscr{B}_\Gamma$. Then

$$(\phi_t)_\sharp \mu = \mu \quad \text{for all } t \in \mathbb{R},$$

where $(\phi_t)_\sharp \mu = \mu \circ \phi_t^{-1}$ is the *pushforward* of μ under ϕ_t.

Exercise 3 (Koopman's Lemma)

Let $\Gamma_* \subset \Gamma$ be a subset of the phase space invariant under the Hamiltonian flow Φ, that is

$$\phi_t \Gamma_* \subset \Gamma_* \quad \text{for all } t \in \mathbb{R}.$$

Let $\mu \in \mathscr{P}(\Gamma_*, \mathscr{B}_{\Gamma_*})$ be a probability measure on Γ_* invariant under the flow Φ, i.e. $(\phi_t)_\sharp \mu = \mu$ for all $t \in \mathbb{R}$.
Define $U_t : L^2(\Gamma_*, \mu) \to L^2(\Gamma_*, \mu)$ by $U_t f = f \circ \phi_t$ for any $t \in \mathbb{R}$ and $f \in L^2(\Gamma_*, \mu)$. Show that $\{U_t : t \in \mathbb{R}\}$ is a unitary group of operators in the Hilbert space $L^2(\Gamma_*, \mu)$.

Exercise 4 (Hard Sphere Collisions)

The phase space for the motion of N hard spheres with diameter ϵ in $\mathbb{R}^d$ is

$$\Gamma = \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN} : |x_i - x_j| \geq \epsilon \text{ if } i \neq j\}.$$

The equations of motion in $\mathring{\Gamma}$ are given by

$$\dot{\boldsymbol{x}} = \boldsymbol{p}, \quad \dot{\boldsymbol{p}} = \boldsymbol{0}.$$

When two particles i and j collide, that is, $|x_i - x_j| = \epsilon$, we interchange the normal components of their momenta,

$$\begin{aligned} p_i' &= p_i - \hat{x} \cdot (p_i - p_j)\, \hat{x} \\ p_j' &= p_j - \hat{x} \cdot (p_j - p_i)\, \hat{x}, \end{aligned}$$

where $\hat{x} = \frac{x_i - x_j}{\epsilon}$ is the unit vector along the line of centres of the spheres (elastic collisions).

(a) Show that elastic collisions of the particles appears in the motion of the phase point $\boldsymbol{z} \in \Gamma$ as elastic reflection from the boundary $\partial\Gamma$ of Γ.

(b) Let $n_1, n_2 \in \mathbb{S}^{d-1}$. Show that the reflection operators $R_j = I - 2n_j \otimes n_j$, $j = 1, 2$, commute if $n_1 \perp n_2$. What does this mean for the particle collisions?