Mathematical Topics in Kinetic Theory

Exercise Sheet 3

Exercise 6 (Bobylev Identity)

Consider the (bilinear) Boltzmann collision operator with collision kernel B,

$$Q(g,f)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left(g(v_*')f(v') - g(v_*)f(v)\right)\,\mathrm{d}\sigma\mathrm{d}v_*.$$

where the σ-representation of the collision process, in which

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \text{for } \sigma\in\mathbb{S}^2,$$

is used.

Theorem 1 (Bobylëv Identity)**.** The Fourier transform of the Boltzmann collision operator is given by

$$\widehat{Q(g,f)}(\xi) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right)\left(\hat{g}(\xi^- + \xi_*)\hat{f}(\xi^+ - \xi_*) - \hat{g}(\xi_*)\hat{f}(\xi-\xi_*)\right)\,\mathrm{d}\xi_*\,\mathrm{d}\sigma,$$

where $\xi^\pm = \frac{\xi\pm|\xi|\sigma}{2}$ and the Fourier transform $\widehat{B}$ of B is with respect to the first argument.

Corollary 2 (Bobylëv Identity for Maxwellian Molecules)**.** Assume that the collision kernel depends only on the deviation angle, i.e. $B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) = b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)$. Then

$$\widehat{Q(g,f)}(\xi) = \int_{\mathbb{S}^{d-1}} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\hat{g}(\xi^-)\hat{f}(\xi^+) - \hat{g}(0)\hat{f}(\xi)\right)\,\mathrm{d}\sigma.$$

Remark 3. We use the following convention for the Fourier transform of a function $f\in\mathcal{S}(\mathbb{R}^d)$:

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} f(v)\,e^{-2\pi i\xi\cdot v}\,\mathrm{d}v.$$

By duality, it can be extended to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ and has the following properties:

(i) $f\in L^1(\mathbb{R}^d)$ implies $\hat{f}\in L^\infty(\mathbb{R}^d)$ with $\|\hat{f}\|_{L^\infty}\le\|f\|_{L^1}$. Furthermore, $\hat{f}$ is continuous and decays at infinity, $\hat{f}(\xi)\overset{|\xi|\to\infty}{\longrightarrow}0$ (Riemann-Lebesgue lemma).

(ii) If $f\in L^1(\mathbb{R}^d)$ is non-negative, $f\ge 0$, one has $\|\hat{f}\|_{L^\infty} = \hat{f}(0) = \|f\|_{L^1}$.

(iii) $f,g\in L^2(\mathbb{R}^d)$ implies $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$ and $\langle f,g\rangle = \langle\hat{f},\hat{g}\rangle$.

(iv) The Fourier transform of the convolution of two functions f, g is given by $\widehat{f * g} = \hat{f}\,\hat{g}$.

(v) For any multiindex $\alpha \in \mathbb{N}_0^d$ one has $\widehat{\partial^\alpha f}(\xi) = (2\pi\mathrm{i}\xi)^\alpha \hat{f}(\xi)$ and $\partial^\alpha \hat{f}(\xi) = \big((-2\pi\mathrm{i}\,\cdot)^\alpha f\big)^{\widehat{}}\,(\xi)$.

(vi) For the Dirac distribution $\delta_a \in \mathcal{S}'(\mathbb{R}^d)$ centered at $a \in \mathbb{R}^d$ one has $\widehat{\delta_a} = e^{-2\pi\mathrm{i}a\cdot}$, in particular $\widehat{\delta_0} = 1$.

See, for instance, sections 2.2–2.4 in L. Grafakos, *Classical Fourier Analysis* (3rd edition), Graduate Texts in Mathematics **249**, Springer New York, 2014.

Proof of Theorem 1. First, assume B to be integrable over $\mathbb{S}^{d-1}$ (Grad's cut-off assumption). The non-cutoff case then follows by a limiting argument, which uses the results of Exercise 8 (weak formulation of the Boltzmann operator).
In the cut-off case, Q can be split into a gain term Q^+ and a loss term Q^-,

$$Q^+(g,f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*')f(v')$$
$$Q^-(g,f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v).$$

For the gain term, a pre-/post-collisional change of variables yields, for suitable test functions φ,

$$\int_{\mathbb{R}^d} Q^+(g,f)(v)\varphi(v)\,\mathrm{d}v = \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\varphi(v')\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma.$$

In particular, choosing $\varphi(v) = e^{-2\pi\mathrm{i}\xi\cdot v}$, we get

$$\widehat{Q^+(g,f)}(\xi) = \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\,e^{-2\pi\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,e^{-2\pi\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma.$$

Bobylëv's observation was that the function

$$(q,\xi) \mapsto \int_{\mathbb{S}^{d-1}} B\left(|q|, \frac{q}{|q|}\cdot\sigma\right) e^{-2\pi\mathrm{i}\frac{|q|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma$$

is an isotropic scalar function, depending only on $|q|$, $|\xi|$ and $\xi\cdot q$. One can therefore interchange the directions of the two vectors (keeping their lengths fixed). In other words, there is an isometry on $\mathbb{S}^{d-1}$ exchanging $\frac{q}{|q|}$ and $\frac{\xi}{|\xi|}$. It follows that

$$\begin{aligned}\widehat{Q^+(g,f)}(\xi) &= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) g(v_*)f(v)\,e^{-2\pi\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,e^{-2\pi\mathrm{i}\frac{|\xi|}{2}(v-v_*)\cdot\sigma}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\\ &= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) g(v_*)f(v)\,e^{-2\pi\mathrm{i}v\cdot\xi^+}\,e^{-2\pi\mathrm{i}v_*\cdot\xi^-}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\end{aligned}$$

Invoking the Fourier inversion formula $g(v_*) = \int_{\mathbb{R}^d}\hat{g}(\eta_*)\,e^{2\pi\mathrm{i}\eta_*\cdot v_*}\,\mathrm{d}\eta_*$, $f(v) = \int_{\mathbb{R}^d}\hat{f}(\eta)\,e^{2\pi\mathrm{i}\eta\cdot v}\,\mathrm{d}v$ yields

$$\widehat{Q^+(g,f)}(\xi) = \int_{\mathbb{R}^{4d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\eta_*)\hat{f}(\eta)\,e^{-2\pi\mathrm{i}v\cdot(\xi^+-\eta)}\,e^{-2\pi\mathrm{i}v_*\cdot(\xi^--\eta_*)}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\eta\,\mathrm{d}\eta_*\,\mathrm{d}\sigma.$$

The coordinate transformation $v_* \mapsto q = v - v_*$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-2\pi i v\cdot(\xi^+-\eta)}\, e^{-2\pi i v_*\cdot(\xi^- -\eta_*)}\, \mathrm{d}v_*\, \mathrm{d}v \\
&= \int_{\mathbb{R}^{2d}} B\left(|q|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-2\pi i v\cdot(\xi^+-\eta+\xi^- -\eta_*)}\, e^{-2\pi i q\cdot(\eta_*-\xi^-)}\, \mathrm{d}q\, \mathrm{d}v \\
&= \widehat{B}\left(|\eta_*-\xi^-|, \frac{\xi}{|\xi|}\cdot\sigma\right) \delta_{\{\eta=\xi-\eta_*\}},
\end{aligned}$$

where $\widehat{B}\left(|\zeta|, \frac{\xi}{|\xi|}\cdot\sigma\right) = \int_{\mathbb{R}^d} B\left(|q|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-2\pi i\zeta\cdot q}\, \mathrm{d}q$ is the Fourier transform of the collision kernel B with respect to the relative velocity variable. Setting $\xi_* = \eta_* - \xi^-$, it follows that

$$\begin{aligned}
\widehat{Q^+(g,f)}(\xi) &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widehat{B}\left(|\eta_*-\xi^-|, \frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\eta_*)\hat{f}(\xi-\eta_*)\, \mathrm{d}\eta_*\, \mathrm{d}\sigma \\
&= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi^- +\xi_*)\hat{f}(\xi^+-\xi_*)\, \mathrm{d}\eta_*\, \mathrm{d}\sigma.
\end{aligned}$$

For the Fourier transform of the loss term,

$$\widehat{Q^-(g,f)}(\xi) = \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\, e^{-2\pi i v\cdot\xi}\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma,$$

one argues analogously that

$$\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) \mathrm{d}\sigma = \int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \mathrm{d}\sigma$$

and therefore

$$\begin{aligned}
\widehat{Q^-(g,f)}(\xi) &= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) g(v_*)f(v)\, e^{-2\pi i v\cdot\xi}\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma \\
&= \int_{\mathbb{R}^{3d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi_*)f(v)\, e^{-2\pi i v\cdot\xi}\, e^{2\pi i v_*\cdot\xi_*}\, \mathrm{d}\xi_*\, \mathrm{d}v\, \mathrm{d}v_*\, \mathrm{d}\sigma.
\end{aligned}$$

Writing $q = v - v_*$ it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} B\left(|v-v_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{2\pi i v_*\cdot\xi_*}\, \mathrm{d}v_* = \int_{\mathbb{R}^d} B\left(|q|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{2\pi i\xi_*\cdot(v-q)}\, \mathrm{d}q \\
&= e^{2\pi i\xi_*\cdot v} \int_{\mathbb{R}^d} B\left(|q|, \frac{\xi}{|\xi|}\cdot\sigma\right) e^{-2\pi i\xi_*\cdot q}\, \mathrm{d}q = e^{2\pi i\xi_*\cdot v}\widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{Q^-(g,f)}(\xi) &= \int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi_*)f(v)\, e^{-2\pi i v\cdot(\xi-\xi_*)}\, \mathrm{d}v\, \mathrm{d}\xi_*\, \mathrm{d}\sigma \\
&= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) \hat{g}(\xi_*)\hat{f}(\xi-\xi_*)\, \mathrm{d}\xi_*\, \mathrm{d}\sigma.
\end{aligned}$$

□

Proof of Corollary 2. In the case $B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) = b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)$ one has

$$\widehat{B}\left(|\xi_*|, \frac{\xi}{|\xi|}\cdot\sigma\right) = \delta_{\{\xi_*=0\}}\, b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)$$

and theorem 1 immediately yields the result. □

References

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