Mathematical Topics in Kinetic Theory

Exercise Sheet 4

Exercise 7 (Symmetrisation of the Boltzmann collision kernel)

The Boltzmann collision operator with collision kernel B is given by

$$Q_B(f,f)(v) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left(f(v_*')f(v') - f(v_*)f(v)\right)\,\mathrm{d}\sigma\,\mathrm{d}v_*.$$

Show that without loss of generality, we can assume that the collision kernel B is supported on the set $0 \le \theta \le \frac{\pi}{2}$, i.e. $(v-v_*)\cdot\sigma \ge 0$.

Hint: Define

$$\overline{B}\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) := \left[B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) + B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot(-\sigma)\right)\right]1_{\{(v-v_*)\cdot\sigma\ge 0\}}$$

and show that $Q_B(f,f) = Q_{\overline{B}}(f,f)$.

Exercise 8 (Momentum transfer and weak formulation for singular collision kernels)

Assume that $B(|z|, k\cdot\sigma) = |z|^\gamma b(\cos\theta)$, where $k = \frac{z}{|z|}$, $\cos\theta = k\cdot\sigma$, $\theta\in[0,\pi/2]$ (see Exercise above), and angular kernel given by

$$\sin^{d-2}\theta\, b(\cos\theta) = \frac{\kappa}{(\sin\theta)^{1+2\nu}}$$

for some $\kappa > 0$, $0 \le \nu < 1$, in dimensions $d \ge 2$.

(a) Show that for all $k \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)\,\mathrm{d}\sigma = \infty,$$

that is, $b \notin L^1([0,1], (1-u^2)^{\frac{d-3}{2}}\,\mathrm{d}u)$, but that

$$\int_{\mathbb{S}^{d-1}} b(k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma < \infty.$$

(b) Define the *cross section for momentum transfer* by[1]

$$\mathcal{M}(|z|) := \int_{\mathbb{S}^{d-1}} B(|z|, k\cdot\sigma)(1-k\cdot\sigma)\,\mathrm{d}\sigma, \quad z\in\mathbb{R}^d.$$

Show that $z \mapsto \mathcal{M}(|z|) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if and only of $\gamma > -d$.

[1] So that $\int_{\mathbb{S}^{d-1}} B(|z|, k\cdot\sigma)(v'-v)\,\mathrm{d}\sigma = \frac{1}{2}(v-v_*)\mathcal{M}(|v-v_*|)$, see also the calculations for part (c).

(c) Let $\mathcal{T}$ be the linear operator

$$\mathcal{T} : \varphi \mapsto \int_{\mathbb{S}^{d-1}} B(|v - v_*|, k \cdot \sigma) \, (\varphi(v') - \varphi(v)) \, \mathrm{d}\sigma, \quad k = \frac{v - v_*}{|v - v_*|}.$$

Show that for all $\varphi \in W^{2,\infty}(\mathbb{R}^d)$,

$$|(\mathcal{T}\varphi)(v, v_*)| \leq \frac{1}{2} \|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)} |v - v_*| \left(1 + \frac{|v - v_*|}{2}\right) \mathcal{M}(|v - v_*|).$$

(d) Conclude that if $\gamma \geq -1$ and $f, g \in L^1_{2+\gamma}(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d) : \|f\|_{L^1_{2+\gamma}} := \int_{\mathbb{R}^d} |f(v)| \langle v \rangle^{2+\gamma} \, \mathrm{d}v < \infty \right\}$, where $\langle v \rangle = (1 + |v|^2)^{1/2}$, then Maxwell's weak formulation

$$\langle Q(g, f), \varphi \rangle = \int_{\mathbb{R}^{2d} \times \mathbb{S}^{d-1}} |v - v_*|^\gamma b \left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) g(v_*) f(v) \, (\varphi(v') - \varphi(v)) \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

is well-definied for all $\varphi \in W^{2\infty}(\mathbb{R}^d)$, with the bound

$$|\langle Q(g, f), \varphi \rangle| \leq C_b \|\varphi\|_{W^{2,\infty}(\mathbb{R}^d)} \|g\|_{L^1_{2+\gamma}} \|f\|_{L^1_{2+\gamma}}.$$