Mathematical Topics in Kinetic Theory

Exercise Sheet 6

Exercise 10 (Conservation of higher moments for the homogeneous Boltzmann equation with bounded collision kernel)

Assume that $0 \le f_0 \in L^1_\kappa(\mathbb{R}^d)$ for some $\kappa \ge 2$ and that B satisfies the assumptions

(i) $0 \le B\left(|v - v_*|, \frac{v-v_*}{|v-v_*|} \cdot \sigma\right) \le C_B$,

(ii) $B(|v - v_*|, \frac{v-v_*}{|v-v_*|} \cdot \sigma) \le K_B(1 + |v|^\lambda + |v_*|^\lambda)$ with $\lambda = \min\{\kappa/2, 2\}$.

Show that the solution of the homogeneous Boltzmann equation constructed in Theorem III.1 satisfies

$$f(t,\cdot) \in L^1_\kappa(\mathbb{R}^d) \quad \text{for all } t \ge 0$$

and for all $0 \le t \le T$ we have the estimate

$$\|f(t,\cdot)\|_{L^1_\kappa} \le c_T \|f_0\|_{L^1_\kappa},$$

where the constant c_T depends only on $\|f_0\|_{L^1_\kappa}$, T, K_B, and κ.

Exercise 11 (Boltzmann H theorem)

Let f be a solution to the homogeneous Boltzmann equation with kernel satisfying

$$0 \le B\left(|v - v_*|, \frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \le C_B$$

and initial datum $0 \le f_0 \in L^1_2(\mathbb{R}^d)$. Assume that, in addition, f has the property that

$$\epsilon_T \, \mathrm{e}^{-C_T|v|^2} \le f(t,v) \le K_T, \quad 0 \le t \le T,$$

for some constants $T, \epsilon_T, C_T, K_T > 0$. Show that for the Boltzmann H functional we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f(t,\cdot)) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} f(t,v) \log f(t,v) \, \mathrm{d}v \le 0.$$