Mathematical Topics in Kinetic Theory

Exercise Sheet 6

Exercise 10 (Conservation of higher moments for the homogeneous Boltzmann equation with bounded collision kernel)

Assume that $0 \le f_0 \in L^1_\kappa(\mathbb{R}^d)$ for some $\kappa \ge 2$ and that B satisfies the assumptions

(i) $0 \le B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) \le \frac{C_B}{|\mathbb{S}^{d-1}|}$,

(ii) $B(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma) \le \frac{K_B}{|\mathbb{S}^{d-1}|}(1+|v|^\lambda+|v_*|^\lambda)$ with $\lambda = \min\{\kappa/2, 2\}$.

Show that the solution of the homogeneous Boltzmann equation constructed in Theorem III.1 satisfies

$$f(t,\cdot) \in L^1_\kappa(\mathbb{R}^d) \quad \text{for all } t \ge 0$$

and for all $0 \le t \le T$ we have the estimate

$$\|f(t,\cdot)\|_{L^1_\kappa} \le c_T \|f_0\|_{L^1_\kappa},$$

where the constant c_T depends only on $\|f_0\|_{L^1_\kappa}$, T, K_B, and κ.

Solution: We first show the following inequality:

Lemma 1. Let $a, b \ge 0$ and $\delta s \le 1 \le s$. Then there exists a constant $\beta_s > 0$ such that

$$a^s + b^s \le (a+b)^s \le a^s + b^s + \beta_s\left(a^{\delta s} b^{(1-\delta)s} + a^{(1-\delta)s} b^{\delta s}\right).$$

Proof. The first inequality follows from convexity of the function $t \mapsto t^s$ for $s \ge 1$. More generally, let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex function with $\Phi(0) \le 0$. Then Φ is superadditive on $[0,\infty)$, that is,

$$\Phi(a) + \Phi(b) \le \Phi(a+b), \quad \text{for all } a, b \ge 0.$$

Indeed, by convexity, for any $\lambda \in [0,1]$ and $x \ge 0$ we have

$$\Phi(\lambda x) = \Phi(\lambda x + (1-\lambda)0) \le \lambda\Phi(x) + (1-\lambda)\Phi(0) \le \lambda\Phi(x),$$

hence for any $a, b \ge 0$,

$$\begin{aligned}\Phi(a) + \Phi(b) &= \Phi\left((a+b)\frac{a}{a+b}\right) + \Phi\left((a+b)\frac{b}{a+b}\right)\\ &\le \frac{a}{a+b}\Phi(a+b) + \frac{b}{a+b}\Phi(a+b) = \frac{a+b}{a+b}\Phi(a+b) = \Phi(a+b).\end{aligned}$$

Remark 2. The above proof can be adapted to concave functions $\Psi : \mathbb{R} \to \mathbb{R}$ with $\Psi(0) \ge 0$. In this case, Ψ is a subadditive function on $[0,\infty)$, that is,

$$\Psi(a+b) \le \Psi(a) + \Psi(b) \quad \text{for all } a, b \ge 0.$$

For the second inequality, we look at

$$(a+b)^s = \left[(a+b)^{\delta s}\right]^{1/\delta} \le \left(a^{\delta s} + b^{\delta s}\right)^{1/\delta}$$

where we used Remark 2 on the concave function $t \mapsto t^{\delta s}$, since $\delta s \le 1$ by assumption. It follows that

$$\begin{aligned}(a+b)^s - a^s - b^s \le \left(a^{\delta s} + b^{\delta s}\right)^{1/\delta} - a^s - b^s &= a^s \left[\left(1 + \left(\frac{b}{a}\right)^{\delta s}\right)^{1/\delta} - 1 - \left(\frac{b}{a}\right)^s\right] \\ &= a^s \left[(1+x)^{1/\delta} - 1 - x^{1/\delta}\right] \quad \text{with } x = \left(\frac{b}{a}\right)^{\delta s}. \qquad (1)\end{aligned}$$

We now distinguish the cases $\frac{1}{\delta} \ge 2$ and $1 \le \frac{1}{\delta} < 2$. (Since $\delta s \le 1 \le s$, we have $\frac{1}{\delta} \ge s \ge 1$).

(i) $\frac{1}{\delta} \ge 2$: By the fundamental theorem of calculus,

$$\begin{aligned}(1+x)^{1/\delta} - 1 - x^{1/\delta} &= \int_0^x \frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1+\xi)^{1/\delta} - \xi^{1/\delta}\right] \mathrm{d}\xi = \frac{1}{\delta}\int_0^x \left[(1+\xi)^{1/\delta-1} - \xi^{1/\delta-1}\right] \mathrm{d}\xi \\ &= \frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\int_0^x \int_\xi^{1+\xi} t^{1/\delta-2}\,\mathrm{d}t\,\mathrm{d}\xi \le \frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\int_0^x (1+\xi)^{1/\delta-2}\,\mathrm{d}\xi \\ &\le \frac{1}{\delta}\left(\frac{1}{\delta}-1\right) x(1+x)^{1/\delta-2} \le \frac{1}{\delta}\left(\frac{1}{\delta}-1\right) x\left(2\max\{1,x\}\right)^{1/\delta-2} \\ &= 2^{1/\delta-2}\frac{1}{\delta}\left(\frac{1}{\delta}-1\right) x \max\left\{1, x^{1/\delta-2}\right\} \le 2^{1/\delta-2}\frac{1}{\delta}\left(\frac{1}{\delta}-1\right) x\left(1 + x^{1/\delta-2}\right) \\ &= 2^{1/\delta-2}\frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\left(x + x^{1/\delta-1}\right).\end{aligned}$$

Inserting this in the bound (1), we obtain

$$(a+b)^s - a^s - b^s \le c_{1/\delta} a^s \left(\frac{b^{\delta s}}{a^{\delta s}} + \left(\frac{b^{\delta s}}{a^{\delta s}}\right)^{1/\delta-1}\right) = c_{1/\delta}\left(a^{(1-\delta)s} b^{\delta s} + b^{(1-\delta)s} a^{\delta s}\right)$$

with $c_{1/\delta} = 2^{1/\delta-2}\frac{1}{\delta}\left(\frac{1}{\delta}-1\right)$.

(ii) $1 \le \frac{1}{\delta} < 2$: Again by the fundamental theorem of calculus, we have

$$(1+x)^{1/\delta} - 1 - x^{1/\delta} = \frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\int_0^x \int_\xi^{1+\xi} t^{1/\delta-2}\,\mathrm{d}t\,\mathrm{d}\xi \le \frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\int_0^x \xi^{1/\delta-2}\,\mathrm{d}\xi = \frac{1}{\delta}x^{1/\delta-1},$$

and therefore

$$(a+b)^s - a^s - b^s \le \frac{1}{\delta} a^s \left(\frac{b^{\delta s}}{a^{\delta s}}\right)^{1/\delta-1} = \frac{1}{\delta} a^{\delta s} b^{(1-\delta)s}.$$

By symmetry, exchanging the roles of a and b, we also have the inequality

$$(a+b)^s - a^s - b^s \le \frac{1}{\delta} b^{\delta s} a^{(1-\delta)s},$$

so adding both inequalities and dividing by 2 yields

$$(a+b)^s - a^s - b^s \le \frac{1}{2\delta}\left(a^{\delta s} b^{(1-\delta)s} + b^{\delta s} a^{(1-\delta)s}\right).$$

This completes the proof of Lemma 1. $\square$

With Lemma 1 at hand, it is now easy to show the following inequality due to POVZNER:

Lemma 3 (POVZNER's inequaltiy). Let $0 \le f_0 \in L^1_\kappa(\mathbb{R}^d)$ for some $\kappa \ge 2$ and assume that B satisfies the assumptions

(i) $0 \le B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) \le \frac{C_B}{|\mathbb{S}^{d-1}|}$,

(ii) $B(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma) \le \frac{K_B}{|\mathbb{S}^{d-1}|}(1+|v|^\lambda+|v_*|^\lambda)$ with $\lambda = \min\{\kappa/2, 2\}$.

Then for the solution of the homogeneous Boltzmann equation constructed in Theorem III.1 we have

$$\int_{\mathbb{R}^d} Q(f,f)(t,v)\,\langle v\rangle^\kappa\,\mathrm{d}v \le \frac{3}{2}K_B\beta_{\kappa/2}\left[\|f\|_{L^1_{\kappa+\lambda-\theta}}\|f\|_{L^1_\theta} + \|f\|_{L^1_{\kappa-\theta}}\|f\|_{L^1_{\lambda+\theta}}\right]$$

for $0 \le \theta \le 2$.[1]

Proof. Lemma 1 apllied to $\langle v'\rangle^\kappa + \langle v'_*\rangle^\kappa$ yields

$$\begin{aligned}\langle v'\rangle^\kappa + \langle v'_*\rangle^\kappa &= (1+|v'|^2)^{\kappa/2} + (1+|v'_*|^2)^{\kappa/2} \le (2+|v'|^2+|v'_*|^2)^{\kappa/2}\\ &= (2+|v|^2+|v_*|^2)^{\kappa/2}\end{aligned}$$

by microscopic conservation of kinetic energy, $|v'|^2+|v'_*|^2 = |v|^2+|v_*|^2$. Again by Lemma 1, we can estimate this quantity further by

$$\begin{aligned}\langle v'\rangle^\kappa + \langle v'_*\rangle^\kappa &\le (1+|v|^2+1+|v_*|^2)^{\kappa/2}\\ &\le (1+|v|^2)^{\kappa/2} + (1+|v_*|^2)^{\kappa/2}\\ &\quad + \beta_{\kappa/2}\left[(1+|v|^2)^{\frac{\kappa-\theta}{2}}(1+|v_*|^2)^{\frac{\theta}{2}} + (1+|v|^2)^{\frac{\theta}{2}}(1+|v_*|^2)^{\frac{\kappa-\theta}{2}}\right]\\ &= \langle v\rangle^\kappa + \langle v_*\rangle^\kappa + \beta_{\kappa/2}\left[\langle v\rangle^{\kappa-\theta}\langle v_*\rangle^\theta + \langle v\rangle^\theta\langle v_*\rangle^{\kappa-\theta}\right]\end{aligned}$$

for $0 \le \theta \le 2$, i.e.,

$$\langle v'\rangle^\kappa + \langle v'_*\rangle^\kappa - \langle v\rangle^\kappa - \langle v_*\rangle^\kappa \le \beta_{\kappa/2}\left[\langle v\rangle^{\kappa-\theta}\langle v_*\rangle^\theta + \langle v\rangle^\theta\langle v_*\rangle^{\kappa-\theta}\right].$$

After a change of coordinates (see Theorem II....) we have

$$\begin{aligned}&\int_{\mathbb{R}^d} Q(f,f)(t,v)\,\langle v\rangle^\kappa\,\mathrm{d}v\\ &= \frac{1}{2}\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left[\langle v'\rangle^\kappa + \langle v'_*\rangle^\kappa - \langle v\rangle^\kappa - \langle v_*\rangle^\kappa\right] f(v)f(v_*)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\\ &\le \frac{\beta_{\kappa/2}}{2}\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left[\langle v\rangle^{\kappa-\theta}\langle v_*\rangle^\theta + \langle v\rangle^\theta\langle v_*\rangle^{\kappa-\theta}\right] f(v)f(v_*)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\end{aligned}$$

and using assumption (ii), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} Q(f,f)(t,v)\,\langle v\rangle^\kappa\,\mathrm{d}v\\ &\le \frac{K_B\beta_{\kappa/2}}{2}\iint_{\mathbb{R}^{2d}}\left(1+|v|^\lambda+|v_*|^\lambda\right)\left[\langle v\rangle^{\kappa-\theta}\langle v_*\rangle^\theta + \langle v\rangle^\theta\langle v_*\rangle^{\kappa-\theta}\right] f(v)f(v_*)\,\mathrm{d}v\mathrm{d}v_*\end{aligned}$$

[1] Recall that $\langle v\rangle = (1+|v|^2)^{1/2}$

Since $\lambda = \min\{\kappa/2, 2\} \in [1, 2]$, we can use convexity once more to obtain for any $x \geq 0$

$$1 + x^\lambda \leq (1+x)^\lambda = \left[(1+x)^2\right]^{\lambda/2} \leq 2^{\lambda/2}(1+x^2)^{\lambda/2} \leq 2(1+x^2)^{\lambda/2},$$

hence

$$1 + |v|^\lambda + |v_*|^\lambda \leq 1 + |v|^\lambda + 1 + |v_*|^\lambda \leq 2\left(\langle v\rangle^\lambda + \langle v_*\rangle^\lambda\right).$$

Notice that we can actually do a bit better here, since for any $y \geq 0$, $y^\lambda = (y^2)^{\lambda/2} \leq (1+y^2)^{\lambda/2}$, so that

$$1 + x^\lambda + y^\lambda \leq 2(1+x^2)^{\lambda/2} + (1+y^2)^{\lambda/2},$$

respectively,

$$1 + x^\lambda + y^\lambda \leq (1+x^2)^{\lambda/2} + 2(1+y^2)^{\lambda/2},$$

so that

$$1 + x^\lambda + y^\lambda \leq \frac{3}{2}\left((1+x^2)^{\lambda/2} + (1+y^2)^{\lambda/2}\right).$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} Q(f,f)(t,v)\,\langle v\rangle^\kappa\,\mathrm{d}v \\
&\leq \frac{3}{2}\frac{K_B\beta_{\kappa/2}}{2}\iint_{\mathbb{R}^{2d}}\left(\langle v\rangle^\lambda + \langle v_*\rangle^\lambda\right)\left[\langle v\rangle^{\kappa-\theta}\langle v_*\rangle^\theta + \langle v\rangle^\theta\langle v_*\rangle^{\kappa-\theta}\right] f(v)f(v_*)\,\mathrm{d}v\mathrm{d}v_* \\
&= \frac{3}{2}\frac{K_B\beta_{\kappa/2}}{2}\iint_{\mathbb{R}^{2d}}\left[\langle v\rangle^{\kappa+\lambda-\theta}\langle v_*\rangle^\theta + \langle v\rangle^{\theta+\lambda}\langle v_*\rangle^{\kappa-\theta} + \langle v_*\rangle^{\kappa+\lambda-\theta}\langle v\rangle^\theta + \langle v_*\rangle^{\theta+\lambda}\langle v\rangle^{\kappa-\theta}\right] f(v)f(v_*)\,\mathrm{d}v\mathrm{d}v_* \\
&= \frac{3}{2}K_B\beta_{\kappa/2}\left(\|f\|_{L^1_{\kappa+\lambda-\theta}}\|f\|_{L^1_\theta} + \|f\|_{L^1_{\theta+\lambda}}\|f\|_{L^1_{\kappa-\theta}}\right).
\end{aligned}$$

□

With Povzner's Lemma at hand, it is now easy to prove the statement of Exercise 10. Indeed, let $\{f^n\}_{n\in\mathbb{N}}$ be the sequence of successive approximations of the solution f of the homogeneous Boltzmann equation with bounded collision kernel as in Theorem II.2.
Assume that $2 \leq \kappa \leq 4$ and that

$$f^{n-1}(t,\cdot) \in L^1_\kappa, \quad t \geq 0,$$

and assume that there exists a constant c_T depending only on $\|f_0\|_{L^1_2}, T, K_B$, and κ, such that

$$\|f^{n-1}(t)\|_{L^1_\kappa} \leq c_T\|f_0\|_{L^1_\kappa}, \quad t \in [0,T].$$

Since $f^1 \equiv 0$, this holds in particular for f^1.
Recall that the successive iterates satisfy

$$\|f^n(t)\|_{L^1} \leq \|f_0\|_{L^1}, \tag{2}$$
$$\|f^n(t)\|_{L^1_2} \leq \|f_0\|_{L^1_2}, \tag{3}$$
$$\partial_t f^n + hf^n = \mathbb{Q}(f^{n-1}), \quad n \in \mathbb{N}, \tag{4}$$

(Theorems III.1 and III.2). Here we used that $\|f_0\|_{L^1_2} = \int_{\mathbb{R}^d} f_0(v)\,\langle v\rangle^2\,\mathrm{d}v \leq \int_{\mathbb{R}^d} f_0(v)\,\langle v\rangle^\kappa\,\mathrm{d}v < \infty$ as, by assumption, $\kappa \geq 2$ and $\|f_0\|_{L^1_\kappa} < \infty$.

Multiplying equation (4) by $(1+|v|_m^2)^{\kappa/2}$, where $|v|_m = \min\{|v|, m\}$, and integrating, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} f^n(t,v)\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v \\
&\quad\le \int_{\mathbb{R}^d} f_0(v)\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v + \int_0^t \int_{\mathbb{R}^d} Q(f^{n-1}, f^{n-1})(s,v)\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v\,\mathrm{d}s \\
&\qquad + \int_0^t \iint_{\mathbb{R}^{2d}} \left[C f^{n-1}(s,v) f^{n-1}(s,v_*) - h f^n(s,v)\right]\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}s.
\end{aligned}
\tag{5}
$$

By (2) and monotonicity of the approximating sequence, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \left[C f^{n-1}(s,v) f^{n-1}(s,v_*) - h f^n(s,v)\right]\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v\,\mathrm{d}v_* \\
&\quad\le \iint_{\mathbb{R}^{2d}} C f^{n-1}(s,v_*)\left[f^{n-1}(s,v) - f^n(s,v)\right]\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v\,\mathrm{d}v_* \le 0.
\end{aligned}
$$

Further, by Povzner's inequality (Lemma 3) with $\theta = \frac{\kappa}{2} \in [1,2]$, and the assumption that $f^{n-1}(t,\cdot) \in L^1_\kappa$, $\|f^{n-1}\|_{L^1_\kappa} \le c_T \|f_0\|_{L^1_\kappa}$, we can estimate

$$
\begin{aligned}
\int_{\mathbb{R}^d} Q(f^{n-1}, f^{n-1})(s,v)\,(1+|v|_m)^{\kappa/2}\,\mathrm{d}v &\le 3K_B \beta_{\kappa/2} \|f^{n-1}\|_{L^1_{\kappa/2+\lambda}} \|f^{n-1}\|_{L^1_{\kappa/2}} \\
&\le 3K_B \beta_{\kappa/2} \|f^{n-1}\|_{L^1_\kappa} \|f^{n-1}\|_{L^1_2} \\
&\le 3K_B \beta_{\kappa/2} c_T \|f_0\|_{L^1_\kappa} \|f_0\|_{L^1_2},
\end{aligned}
$$

where in the last step we also used (3).
Plugging the above two bounds into (5), we thus have with $0 \le t \le T$,

$$
\begin{aligned}
\|f^n\|_{L^1_\kappa} &\le \|f_0\|_{L^1_\kappa} + T\left(3K_B \beta_{\kappa/2} c_T \|f_0\|_{L^1_\kappa} \|f_0\|_{L^1_2}\right) + 0 \\
&= \left(1 + 3T K_B \beta_{\kappa/2} c_T \|f_0\|_{L^1_2}\right) \|f_0\|_{L^1_\kappa},
\end{aligned}
$$

that is, there exists a constant $\widetilde{c}_T$ depending only on $\|f_0\|_{L^1_2}, T, K_B$, and κ, such that

$$
\|f^n(t,\cdot)\|_{L^1_\kappa} \le \widetilde{c}_T \|f_0\|_{L^1_\kappa}, \quad t \in [0,T].
$$

By induction, this proves the result for $\kappa \in [2,4]$.
For $\kappa \in [4,6]$, we just repeat the above proof, taking $\theta = 2$ in Povzner's inequality. Inductively, this yields the result for all $\kappa \ge 2$.

Exercise 11 (Boltzmann H theorem)
Let f be a solution to the homogeneous Boltzmann equation with kernel satisfying

$$0 \le B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) \le C_B$$

and initial datum $0 \le f_0 \in L^1_2(\mathbb{R}^d)$. Assume that, in addition, f has the property that

$$\epsilon_T\, \mathrm{e}^{-C_T|v|^2} \le f(t,v) \le K_T, \quad 0 \le t \le T, \tag{6}$$

for some constants $T, \epsilon_T, C_T, K_T > 0$. Show that for the Boltzmann H functional we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f(t,\cdot)) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f(t,v)\log f(t,v)\,\mathrm{d}v \le 0.$$

Solution: Let f be the solution of the homogeneous Boltzmann equation with bounded collision kernel and initial datum $0 \le f_0 \in L^1_2 \cap L\log L$. This solution exists for any $t > 0$ and is unique by Theorem III.2 of the lecture. Now assume in addition that the bounds

$$\epsilon_T\, \mathrm{e}^{-C_T|v|^2} \le f(t,v) \le K_T,$$

hold on some time interval $0 \le t \le T$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H(f(t,\cdot)) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} f(t,v)\log f(t,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} (\partial_t f)(t,v)\left(\log f(t,v)+1\right)\,\mathrm{d}v\\
&= \int_{\mathbb{R}^d} Q(f,f)(t,v)\left(\log f(t,v)+1\right)\,\mathrm{d}v\\
&= \int_{\mathbb{R}^d} Q(f,f)(t,v)\log f(t,v)\,\mathrm{d}v + \int_{\mathbb{R}^d} Q(f,f)(t,v)\,\mathrm{d}v
\end{aligned}$$

By the conservation properties of the Boltzmann equation ($\varphi = 1$ is a collisional invariant), the second integral is equal to zero, see also the proof of Theorems III.1 and III.2. So it remains to show that the term $\int_{\mathbb{R}^d} Q(f,f)(t,v)\log f(t,v)\,\mathrm{d}v$ is non-negative. Let us first check, however, that it is well-defined, i.e. $Q(f,f)(t,v)\log f(t,v) \in L^1(\mathbb{R}^d_v)$ for all $T \ge 0$, so that all the changes of coordinates discussed in Chapter II of the lecture work. Indeed, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|Q(f,f)(t,v)\right| \left|\log f(t,v)\right|\,\mathrm{d}v\\
&\le \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left|f(t,v_*')f(t,v') - f(t,v_*)f(t,v)\right|\left|\log f(t,v)\right|\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\\
&\le \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left[f(t,v_*')f(t,v') + f(t,v_*)f(t,v)\right]\left|\log f(t,v)\right|\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma
\end{aligned}$$

where we used non-negativity of B and f. By the bounds (6) we have

$$\log\epsilon_T - C_T|v|^2 \le \log f(t,v) \le \log K_T,$$

hence

$$\left|\log f(t,v)\right| \le \max\{|\log K_T|, |\log\epsilon_T - C_T|v|^2|\} \le c_T(1+|v|^2)$$

for some suitably large $c_T > 0$.

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|Q(f,f)(t,v)\right| \left|\log f(t,v)\right| \, \mathrm{d}v \\
&\leq c_T \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) f(t,v_*')f(t,v')\,(1+|v|^2)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\
&\quad + c_T \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) f(t,v_*)f(t,v)\,(1+|v|^2)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma
\end{aligned} \tag{7}$$

Let us treat the two terms on the right hand side of inequality (7) separately. In the first term we can do a pre-post-collisional change of coordinates to obtain

$$\begin{aligned}
&\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) f(t,v_*')f(t,v')\,(1+|v|^2)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\
&= \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) f(t,v_*)f(t,v)\,(1+|v'|^2)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\
&\leq \frac{C_B}{|\mathbb{S}^{d-1}|} \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} f(t,v_*)f(t,v)\,(1+|v'|^2)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\
&= \frac{C_B}{|\mathbb{S}^{d-1}|} \iint_{\mathbb{R}^{2d}} f(t,v_*)f(t,v) \int_{\mathbb{S}^{d-1}} (1+|v'|^2)\,\mathrm{d}\sigma\,\mathrm{d}v\mathrm{d}v_*
\end{aligned}$$

Observe that since $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$,

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} (1+|v'|^2)\,\mathrm{d}\sigma &= \int_{\mathbb{S}^{d-1}} \left(1 + \frac{v^2+v_*^2}{2} - \frac{v\cdot v_*}{2} + \frac{|v-v_*|(v-v_*)}{2}\cdot\sigma\right)\mathrm{d}\sigma \\
&= |\mathbb{S}^{d-1}|\left(1+\frac{v^2+v_*^2}{2} - \frac{v\cdot v_*}{2}\right)
\end{aligned}$$

where in the last step we used that by symmetry $a\cdot\int_{\mathbb{S}^{d-1}}\sigma\,\mathrm{d}\sigma = 0$ for any $a\in\mathbb{R}^d$. By Cauchy-Schwartz, we can further estimate

$$\int_{\mathbb{S}^{d-1}} (1+|v'|^2)\,\mathrm{d}\sigma \leq |\mathbb{S}^{d-1}|\left(1+v^2+v_*^2\right).$$

With this we get

$$\begin{aligned}
&\frac{C_B}{|\mathbb{S}^{d-1}|}\iint_{\mathbb{R}^{2d}} f(t,v_*)f(t,v)\int_{\mathbb{S}^{d-1}}(1+|v'|^2)\,\mathrm{d}\sigma\,\mathrm{d}v\mathrm{d}v_* \\
&\quad\leq C_B\iint_{\mathbb{R}^{2d}} f(t,v_*)f(t,v)(1+v^2+v_*^2)\,\mathrm{d}v\mathrm{d}v_* \\
&\quad\leq 2C_B\int_{\mathbb{R}^d} f(t,v_*)(1+v_*^2)\,\mathrm{d}v_*\int_{\mathbb{R}^d} f(t,v)\,\mathrm{d}v \\
&\quad= 2C_B\int_{\mathbb{R}^d} f_0(v_*)(1+v_*^2)\,\mathrm{d}v\int_{\mathbb{R}^d} f_0(v)\,\mathrm{d}v = 2C_B\|f_0\|_{L^1_2(\mathbb{R}^d)}\|f_0\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Note that the last step holds because of conservation of mass and kinetic energy for the solutions constructed in Theorem III.2.

For the second term on the right hand side of (7) we just estimate

$$\begin{aligned}
&\iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) f(t,v_*)f(t,v)\,(1+|v|^2)\,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\
&\leq C_B\iint_{\mathbb{R}^{2d}} f(t,v_*)f(t,v)(1+v^2)\,\mathrm{d}v\mathrm{d}v_* = C_B\int_{\mathbb{R}^d} f(t,v_*)\,\mathrm{d}v_*\int_{\mathbb{R}^d} f(t,v)(1+v^2)\,\mathrm{d}v \\
&= C_B\|f_0\|_{L^1(\mathbb{R}^d)}\|f_0\|_{L^1_2(\mathbb{R}^d)},
\end{aligned}$$

again by conservation laws. Altogether, we have thus proved the bound

$$\int_{\mathbb{R}^d} \left|Q(f,f)(t,v)\right| \left|\log f(t,v)\right| \,\mathrm{d}v \leq 3c_T C_B \|f_0\|_{L^1} \|f_0\|_{L^1_2}$$

uniformly for all times $0 \leq t \leq T$.
We can therefore use the pre-post-collisional change of variables and symmetries of the collision kernel to rewrite the integral $\int Q(f,f) \log f \,\mathrm{d}v$ in the fully symmetric form (see Lemma II.????)

$$\begin{aligned}
&\int_{\mathbb{R}^d} Q(f,f)(t,v)\, \log f(t,v)\,\mathrm{d}v \\
&= -\frac{1}{4} \iiint_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} B\left(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma\right) \left[f(v'_*)f(v') - f(v_*)f(v)\right] \log\frac{f(v'_*)f(v')}{f(v_*)f(v)} \,\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \leq 0
\end{aligned}$$

since $B \geq 0$ and the map $(x,y) \mapsto (x-y)(\log x - \log y)$ is non-negative by monotonicity of the logarithm.
Putting everything together, we have thus proved that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(f(t,\cdot)) = \int_{\mathbb{R}^d} Q(f,f)(t,v)\, \log f(t,v)\,\mathrm{d}v \leq 0.$$