Mathematical Methods of Quantum Mechanics
1st Exercise Sheet

1. Let \( H \) be a Hilbert space. Prove the following statements:
   (i) A sequence \((u_n)\) in \( H \) converges to \( u \in H \) if and only if
   \[
   \lim_{n \to \infty} \langle u_n, \phi \rangle = \langle u, \phi \rangle \quad (\phi \in H)
   \]
   and \( \lim_{n \to \infty} \|u_n\| = \|u\| \). Give an example to show that the condition (*) alone does not imply convergence of \((u_n)\) in \( H \).
   (ii) If \( M \) is a subspace of \( H \), then \( (M^\perp)^\perp = M \).

2. Let \( A, B \) be bounded operators on the Hilbert space \( H \). Show that
   (i) \((A + B)^* = A^* + B^*\), \((\lambda A)^* = \overline{\lambda} A^*\) for \( \lambda \in \mathbb{C} \).
   (ii) \((AB)^* = B^* A^*\)
   (iii) \(\|A\| = \|A^*\|\)
   (iv) \(A^{**} = A\)
   (v) \(\|AA^*\| = \|A^*A\| = \|A\|^2\)
   (vi) \(\ker A = (\ran A^*)^\perp, \ker A^* = (\ran A)^\perp\).

3. Let
   \[
   D := \text{linear hull of } \{e^{-|x|^2/2}x_1^{k_1}x_2^{k_2}x_3^{k_3} : k_1 \in \mathbb{N}_0\} \subset L^2(\mathbb{R}^3).
   \]
   (i) Show that \( x \mapsto \phi(x)e^{-|x|^2/2} \in L^1(\mathbb{R}^3) \) for any \( \phi \in L^2(\mathbb{R}^3) \).
   (ii) Let \( \phi \in L^2(\mathbb{R}^3) \) be such that \( \int_{\mathbb{R}^3} \phi(x)e^{-|x|^2/2}x_1^{k_1}x_2^{k_2}x_3^{k_3}dx = 0 \) for all \((k_1, k_2, k_3) \in \mathbb{N}_0^3\). Prove that
   \[
   \int_{\mathbb{R}^3} \phi(x)e^{-|x|^2/2}e^{i\xi \cdot x}dx = 0 \quad (\xi \in \mathbb{R}^3)
   \]
   holds.
   Hint: use the usual series \( e^z = \sum_{n=0}^{\infty} \frac{1}{n!}z^n \) for the exponential function.
   (iii) Show that \( D \) is dense in \( L^2(\mathbb{R}^3) \).
   Hint: Fourier transform.