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Mathematical Methods of Quantum Mechanics
2nd Exercise Sheet

4.

(i) Show that $\|AB\| \leq \|A\|\|B\|$ for bounded operators A, B on a Hilbert space H.

(ii) Let A be a bounded self-adjoint operator. Show that $\|A^{2^m}\| = \|A\|^{2^m}$ ($m \in \mathbb{N}$).

(iii) Let P, Q be bounded self-adjoint operators on the Hilbert space $H \neq \{0\}$. Prove that then the relation

$$[P, Q] = 1$$

for the commutator cannot hold true. What are the consequences for the mathematical structure of quantum mechanics?

Hint: Show first that $PQ^n - Q^nP = nQ^{n-1}$ ($n \in \mathbb{N}$).

5. Let A, B be bounded operators on the Hilbert space H.

(i) Prove Dyson's formula:

$$e^{(A+B)t} = e^{At}\left(1 + \sum_{n=1}^{\infty} \int_0^t \int_0^{t_2} \dots \int_0^{t_{n-1}} B(t_1) \cdots B(t_n) dt_n \dots dt_1\right)$$

where $B(t) := e^{-At}Be^{At}$ and the infinite sum converges in operator norm.

Hint: Differentiate $Z(t) = e^{-At}e^{(A+B)t}$ with respect to t and show that Z satisfies the integral equation

$$Z(t) = 1 + \int_0^t B(t_1)Z(t_1)dt_1.$$

Use the method of successive approximations.

(ii) Prove Trotter's product formula:

$$e^{A+B} = \lim_{n\to\infty} (e^{A/n}e^{B/n})^n$$

(limit to be taken in operator norm).

Hint: Let $S_n = e^{(A+B)/n}, T_n = e^{A/n}e^{B/n}$ and note the following relation (telescopic sum)

$$S_n^n - T_n^n = S_n^n - T_nS_n^{n-1} + T_nS_n^{n-1} + \dots + T_n^{n-1}S_n - T_n^{n-1}S_n - T_n^n.$$

Show $\|S_n - T_n\| = O(\frac{1}{n^2})$.