Karlsruhe Institute of Technology (KIT) SS 2011
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Mathematical Methods of Quantum Mechanics
3rd Exercise Sheet

6. Let $m : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $m \in L^\infty(B)$ for any bounded open set $B \subset \mathbb{R}^d$. Moreover, let

$$D := \{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |m(x)u(x)|^2 dx < \infty\}$$

and $A : D \to L^2(\mathbb{R}^d), (Au)(x) = m(x)u(x)$. Show that D is dense in $L^2(\mathbb{R}^d)$ and that A is self-adjoint.

7. Recall that a continuous function $\psi : [0, 1] \to \mathbb{C}$ is absolutely continuous if and only if

$$\psi(x) = \psi(0) + \int_0^x g(y)dy$$

for some $g \in L^1(0, 1)$. Then $\psi'(x)$ exists a.e. on $(0, 1)$ and equals g a.e.. The space of absolutely continuous functions on $[0, 1]$ is denoted by $AC[0, 1]$.

(i) Consider the operator $A_0 : D(A_0) \to L^2(0, 1)$ defined by $A_0u = iu'$ on

$$D(A_0) := \{u \in AC[0, 1] : u' \in L^2(0, 1), u(0) = u(1) = 0\}.$$

Find out whether A_0 is closed/symmetric/self-adjoint.

(ii) For $\alpha \in \mathbb{R}$ fixed, consider the operator $A_1 : D(A_1) \to L^2(0, 1)$ defined by $A_1u = iu'$ on

$$D(A_1) := \{u \in AC[0, 1] : u' \in L^2(0, 1), u(1) = e^{i\alpha}u(0)\}.$$

What can you say about the self-adjointness of A_1? What is the relation between A_0 and A_1?