10. Let $H$ be a symmetric, semibounded operator in the Hilbert space $H$ (with domain $D(H)$), and $A$ an operator with $D(H) \subseteq D(A)$. Suppose that there exist $\tilde{a}, \tilde{b}$ such that

\[ ||A\phi||^2 \leq \tilde{a}^2 ||H\phi||^2 + \tilde{b}^2 ||\phi||^2 \quad (\phi \in D(H)). \]

Show that then the condition (*) from exercise 9 holds:

\[ ||A\phi|| \leq a ||H\phi|| + b ||\phi|| \quad (\phi \in D(H)) \]

and that

\[ \inf\{a : (*) \text{ holds for some } b > 0\} = \inf\{\tilde{a} : (**) \text{ holds for some } \tilde{b} > 0\}. \]

11. (Magnetic Schrödinger Hamiltonian.) The magnetic Schrödinger Hamiltonian is formally given by

\[ H\phi = (-i\nabla - eA)^2\phi + V\phi \]

with electric charge $e$ and magnetic vector potential $A$. Show that in case of $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and $A \in L^4(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $\nabla \cdot A \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, the Hamiltonian is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ when written in the form

\[ H\phi = -\Delta \phi - 2ieA \cdot \nabla \phi - ie(\nabla \cdot A)\phi + e^2(A \cdot A)\phi + V\phi. \]

Hint: If the exercise seems too difficult, try to take $A \in L^\infty(\mathbb{R}^3), \nabla \cdot A \in L^\infty(\mathbb{R}^3)$. 