Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Prof. Dr. Dirk Hundertmark
Dr. Vu Hoang

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Mathematical Methods of Quantum Mechanics
6th Exercise Sheet

12. Let $f \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

(a) Recall that $p = -i\nabla$ is the momentum operator. Show that

$$(f(p)\phi)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (\mathcal{F}^{-1} f)(x-y)\phi(y)dy \quad (\phi \in L^2(\mathbb{R}^d)),$$

where $\mathcal{F}^{-1} f$ is the inverse Fourier transform of f.

(b) Let $g \in L^2(\mathbb{R}^d)$. Show that the operator $\phi \mapsto g(x)f(p)\phi$ is compact as an operator from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$.

13. In this exercise we consider one of the components of angular momentum operator: $L_3 = x_1 p_2 - x_2 p_1$. Let

$$D := \text{linear hull of} \{e^{-|x|^2/2} x_1^{k_1} x_2^{k_2} x_3^{k_3} : k_i \in \mathbb{N}_0\} \subset L^2(\mathbb{R}^3).$$

(a) Show that the operators $\{R_\varphi\}$, $R_\varphi : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$,

$$(R_\varphi u)(x_1, x_2, x_3) = u(x_1 \cos\varphi - x_2 \sin\varphi,\ x_2 \cos\varphi + x_1 \sin\varphi, x_3) \quad (u \in L^2(\mathbb{R}^3))$$

form a strongly continuous unitary group.

(b) Prove that $\{R_\varphi\}$ is strongly differentiable on D.

(c) Show that L_3 is essentially self-adjoint on D.
Hint: Stone's theorem. What is the connection between the rotation group and L_3?