Problems

1. Show that the following equality holds

\[(f_1 \wedge f_2 \wedge \ldots \wedge f_N)(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^\pi \prod_{j=1}^N f_{\pi(j)}(x_j).\]

2. Assume that \(f_i, i \in \{1, 2, \ldots, N\}\) are orthonormal functions in \(L^2(\mathbb{R}^d)\). Then the following holds,

\[\|f_1 \wedge f_2 \wedge \ldots \wedge f_N\|_{L^2(\mathbb{R}^{dN})} = 1.\]

3. Let \(f_i, g_j \in L^2(\mathbb{R}^d), i, j \in \{1, 2, \ldots, N\}\), then the following holds

\[(f_1 \wedge f_2 \wedge \ldots \wedge f_N, g_1 \wedge g_2 \wedge \ldots \wedge g_N) = \det([f_i, g_j]).\]

4. Let \(f_i \in L^2(\mathbb{R}^d), i \in \{1, 2, \ldots, N\}\) and \(A_{ij}\) be an element of \(N \times N\)-matrix. We define the new set of functions as

\[g_i = \sum_{j=1}^N A_{ij} f_j.\]

Prove that

\[g_1 \wedge g_2 \wedge \ldots \wedge g_N = \det(A)(f_1 \wedge f_2 \wedge \ldots \wedge f_N).\]

In particular if \(A\) is a unitary transformation within the subspace \([f_1, f_2, \ldots f_n]_\lambda\), the wedge product is invariant up to the phase factor.

5. Prove that the following claims are equivalent:
   (a) \(f_1 \wedge f_2 \wedge \ldots \wedge f_N = 0,\)
   (b) \(f_i \in L^2(\mathbb{R}^d), i \in \{1, 2, \ldots, N\}\) are linearly dependent.

Remark: Try to prove one direction using indirect proof (proof by contraposition) and previous results 2 and 4.

Notation

- Antisymmetrized tensor product of functions \(f_1, f_2, \ldots, f_N\)

\[(f_1 \wedge f_2 \wedge \ldots \wedge f_N)(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} (-1)^\pi \prod_{j=1}^N f_j(x_{\pi(j)})\]