Mathematical Methods of Quantum Mechanics 2
Exercise 5

Problems

1. **Newton’s Theorem** Let $\mu$ be a spherically symmetric charge distribution, i.e. $\mu(A) = \mu(R(A))$ for all subsets $A \subset \mathbb{R}^3$ and any rotation $R \in SO(3)$. Then the following holds

$$\int_{\mathbb{R}^3} \frac{1}{|x - y|} d\mu(y) = \frac{1}{|x|} \int_{|y| \leq |x|} d\mu(y) + \int_{|y| > |x|} \frac{1}{|y|} d\mu(y).$$

2. **Stability of matter** Show that the hydrogen-type atom is stable, i.e. the corresponding non-relativistic Hamiltonian

$$H = -\frac{1}{2} \Delta - \frac{Z \alpha}{|x|}.$$

is lower semibounded on $L^2(\mathbb{R}^d)$ for suitable $d$ in the following steps:

(a) find suitable $C_1, C_2$ in the following inequality

$$\left( \psi, \frac{Z \alpha}{|x|} \psi \right) \leq C_1 \| \psi \|^2_{\frac{2d}{d-2}} + C_2 \| \psi \|^2,$$

(b) find suitable $C_3, C_4, C_5$ in the following inequality

$$(\psi, H \psi) \geq C_3 t^2 + C_4 t^{\frac{d-2}{2}} + C_5,$$

for $\| \psi \|_2 = 1$ and $t > 0$,

(c) for $\| \psi \|_2 = 1$ find a lower bound for the expression

$$(\psi, H \psi).$$

3. **Relativistic case** How the situation changes in Problem 2 for the relativistic case?

*Remark*: It is enough to deal with ultrarelativistic case due to the following

$$|p| \leq \sqrt{p^2 + m^2} - m \leq |p| - m.$$

Side notes

- **Sobolev Inequality**
  For any function in $\psi \in \mathcal{H}^1(\mathbb{R}^d), d \geq 3$ the following holds

$$2T_\psi = \| \nabla \psi \|^2 \geq S_d \| \psi \|^2_{\frac{2d}{d-2}}.$$

Furthermore for $d = 3$ we have $S_3 = \frac{3}{4}(4\pi^2)^{\frac{3}{2}}$.

- **Sobolev Inequality for a relativistic particle**
  For any function in $\psi \in \mathcal{H}^1(\mathbb{R}^d), d \geq 2$ the following holds

$$T^R_{\psi} = (\psi, |p| \psi) \geq S'_d \| \psi \|^2_{\frac{2d}{d-2}}.$$

Furthermore for $d = 3$ we have $S'_3 = (2\pi^2)^{\frac{3}{4}}$.

- **Newton’s Shell Theorem**
  For a surface measure $d\sigma(x)$ on a sphere the following holds

$$\int_{|x|=r} \frac{d\sigma(x)}{|x - R|} = \frac{4\pi r^2}{\max\{r, |R|\}},$$

for $R \in \mathbb{R}^3$. 