Mathematical Methods of Quantum Mechanics
Exercise 1

Problems

1. Suppose $\mathfrak{D}$ is a vector space. Let $s(f,g)$ be a sesquilinear form on $\mathfrak{D}$ and $q(f) = s(f,f)$ the associated quadratic form. Prove the *parallelogram law*
$$q(f+g) + q(f-g) = 2q(f) + 2q(g) \tag{1}$$
and the *polarization identity*
$$s(f,g) = \frac{1}{4}\left[q(f+g) - q(f-g)\right] + \frac{i}{4}\left[q(f-ig) - q(f+ig)\right] . \tag{2}$$
Show that $s(f,g)$ is symmetric if and only if $q(f)$ is real-valued.

2. Let $A \in \mathfrak{L}(X)$ be a bijection(one-to-one and onto) from X to X. Show
$$\|A^{-1}\|^{-1} = \inf_{f\in X, \|f\|=1} \|Af\| . \tag{3}$$

3. An operator $A : \mathcal{X} \to \mathcal{X}$ is bounded invertible if it is bijective and A^{-1} is bounded. Let $Inv(\mathcal{X})$ be the set of all bounded invertible operators $A \in \mathfrak{L}(\mathcal{X})$. Show that the set $Inv(\mathcal{X})$ is an open set in $\mathfrak{L}(\mathcal{X})$.
Hint: Make an estimate for the difference of two operators $A-B$ based on Neumann series for the operators.

4. Show that the following statements are equivalent
   (a) $A \in \mathfrak{L}(X)$ is closable,
   (b) $\overline{\Gamma(A)}$ is a graph of an operator,
   (c) $\overline{\Gamma(A)} \ni (0,y)$ implies $y = 0$,
   (d) $(x,y_1),(x.y_2) \in \overline{\Gamma(A)}$ implies $y_1 = y_2$.

Theoretical Background

- **Sesquilinear Form**
Sesquilinear form $s(\cdot,\cdot)$ is a bilinear function from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{C}$, where $\mathcal{H}$ is a Hilbert space (complete vector space with a scalar product), which satisfies:
$$s(x, \alpha y + \beta z) = \alpha s(x, y) + \beta s(x, z) \tag{4}$$
$$s(\alpha x + \beta y, z) = \overline{\alpha} s(x, z) + \overline{\beta} s(y, z) \tag{5}$$
for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

- **Symmetric Sesquilinear Form**
Sesquilinear form $s(\cdot,\cdot)$ is symmetric if
$$s(f, g) = \overline{s(g, f)} \tag{6}$$
holds.

- **Set of all Bounded Linear Operators**
Set of all bounded linear operators from X to Y is denoted by $\mathfrak{L}(X, Y)$. We write $\mathfrak{L}(X, X) = \mathfrak{L}(X)$.

- **Neumann Series**
Let $T \in \mathfrak{L}(X)$. If Neumann Series $\sum_{k=0}^{\infty} T^k$ converges in the operator norm then the inverse of $I - T$ exists and
$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \tag{7}$$
holds.

- **Closable Operator**
An operator T is closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure, which we denote by $\overline{T}$.