Mathematical Methods of Quantum Mechanics
Exercise 1

Problems

1. Suppose $\mathfrak{D}$ is a vector space. Let $s(f, g)$ be a sesquilinear form on $\mathfrak{D}$ and $q(f) = s(f, f)$ the associated quadratic form. Prove the parallelogram law
   \[ q(f + g) + q(f - g) = 2q(f) + 2q(g) \]  
   (1)
   and the polarization identity
   \[ s(f, g) = \frac{1}{4} [q(f + g) - q(f - g)] + \frac{i}{4} [q(f - ig) - q(f + ig)]. \]  
   (2)
   Show that $s(f, g)$ is symmetric if and only if $q(f)$ is real-valued.

2. Let $A \in \mathcal{L}(X)$ be a bijection (one-to-one and onto) from $X$ to $X$. Show
   \[ \|A^{-1}\|^{-1} = \inf_{f \in X, \|f\|=1} \|Af\|. \]  
   (3)

3. An operator $A : \mathcal{X} \to \mathcal{X}$ is bounded invertible if it is bijective and $A^{-1}$ is bounded. Let $\text{Inv}(\mathcal{X})$ be the set of all bounded invertible operators $A \in \mathcal{L}(\mathcal{X})$. Show that the set $\text{Inv}(\mathcal{X})$ is an open set in $\mathcal{L}(\mathcal{X})$.
   Hint: Make an estimate for the difference of two operators $A - B$ based on Neumann series for the operators.

4. Show that the following statements are equivalent
   (a) $A \in \mathcal{L}(X)$ is closable,
   (b) $\overline{\Gamma(A)}$ is a graph of an operator,
   (c) $\overline{\Gamma(A)} \ni (0, y)$ implies $y = 0$,
   (d) $(x, y_1), (x, y_2) \in \overline{\Gamma(A)}$ implies $y_1 = y_2$. 

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Theoretical Background

• **Sesquilinear Form**
  Sesquilinear form $s(\cdot, \cdot)$ is a bilinear function from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$, where $\mathcal{H}$ is a Hilbert space (complete vector space with a scalar product), which satisfies:

$\begin{align*}
  s(x, \alpha y + \beta z) &= \alpha s(x, y) + \beta s(x, z) \\
  s(\alpha x + \beta y, z) &= \overline{\alpha} s(x, z) + \overline{\beta} s(y, z)
\end{align*}$

for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

• **Symmetric Sesquilinear Form**
  Sesquilinear form $s(\cdot, \cdot)$ is symmetric if

$\begin{align*}
  s(f, g) &= \overline{s(g, f)}
\end{align*}$

holds.

• **Set of all Bounded Linear Operators**
  Set of all bounded linear operators from $X$ to $Y$ is denoted by $\mathfrak{L}(X, Y)$. We write $\mathfrak{L}(X, X) = \mathfrak{L}(X)$.

• **Neumann Series**
  Let $T \in \mathfrak{L}(X)$. If Neumann Series $\sum_{k=0}^{\infty} T^k$ converges in the operator norm then the inverse of $I - T$ exists and

$\begin{align*}
  (I - T)^{-1} &= \sum_{k=0}^{\infty} T^k
\end{align*}$

holds.

• **Closable Operator**
  An operator $T$ is closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure, which we denote by $\overline{T}$. 