Mathematical Methods of Quantum Mechanics
Exercise 10

Problems

1. Let the operator $H_0 = -\Delta$ be defined on $H^2(\mathbb{R}^n)$. Furthermore we define the operators for momentum and position in the standard way. Prove the following:
   (a) $-\Delta = \frac{2}{|x|^2} x \cdot \nabla - \frac{1}{|x|^2}(\nabla \cdot x)(x \cdot \nabla) + \frac{1}{|x|^2} \sum_{l<m} L_{l,m}^2$,
   where $L_{l.m} = x_l p_m - x_m p_l$.
   (b) $(\nabla\psi, \nabla\psi) = (\partial_r \psi, \partial_r \psi) + \left(\psi, \frac{1}{r^2} \sum_{l<m} L_{l,m}^2 \psi\right) = (\partial_r \psi, \partial_r \psi) + \left(\frac{\nabla_\theta \psi}{r}, \frac{\nabla_\theta \psi}{r}\right)$,
   where the second term corresponds to the Laplace-Bertrami operator on the unit sphere and $r\partial_r = (x \cdot \nabla)$ with $r = |x|$.
2. Let H_0 be the free Laplacian on $L^2(\mathbb{R}^n)$. Show that the inequality

$$\|H_0^{1/2}\psi\| \leq \|(\beta H_0 + (4\beta)^{-1})\psi\| \leq \beta\|H_0\psi\| + (4\beta)^{-1}\|\psi\|$$

   is valid for any $\beta > 0$ and an arbitrary $\psi \in \mathcal{D}(H_0)$.
3. Consider the Hamiltonian $H = -\Delta + C|x|^2$ on $H^2(\mathbb{R}^n)$ with $C > 0$. From class we know that it has only discrete spectrum. Let E be a discrete eigenvalue with an eigenvector $\psi \in H^2(\mathbb{R}^n)$. Find a good choice of a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $x \to e^{g(|x|)}\psi(x) \in L^2(\mathbb{R}^n)$.
   Hint: Use the weight $F_\epsilon(x) = \frac{g(|x|)}{1+\epsilon g(|x|)}$ and mimic the proof from the lecture.
4. Consider the situation from the previous exercise with the potential $V(x) = C|x|^r$ with $C, r > 0$. Find an optimal estimate on the function $g_r : \mathbb{R}^+ \to \mathbb{R}^+$ with respect to r.