Mathematical Methods of Quantum Mechanics
Exercise 11

Problems

1. Suppose A is closed and R is bounded operator on a Hilbert space $\mathcal{H}$. Show that $R = R_A(z)$ if only if
$$((A-z)^*\phi, R\psi) = (\phi, \psi)$$
for all $\phi \in \mathcal{D}(A^*)$, $\psi \in \mathcal{H}$.
Hint: Show that $R\psi \in \mathcal{D}(A)$ using the closedness of A.

2. Show that for a fixed $x \in \mathbb{R}$ the quadratic form $\psi \to |\psi(x)|^2$ is form bounded by the quadratic form corresponding to the free Laplacian $\|\nabla\psi\|^2$ on $H^1(\mathbb{R})$.

3. Using the previous result show that the quadratic form $(\psi, V\psi)$ with $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ is infinitesimally form bounded with respect to the quadratic form corresponding to the free Laplacian $\|\nabla\psi\|^2$ on $H^1(\mathbb{R})$.

4. Show that the quadratic form $|\psi(0_+) - \psi(0_-)|^2$ is form bounded by the quadratic form corresponding to the free Laplacian $\|\nabla\psi\|^2$ on $H^1(\mathbb{R} \setminus \{0\}) := H^1(\mathbb{R}^-) \oplus H^1(\mathbb{R}^+)$.

5. Let $\psi \in H^1(\mathbb{R}^d)$ with $d \geq 2$. Show that the following holds
   (a) $\|\psi\|_q \leq \|\hat{\psi}\|_p \leq C(\||\eta|\hat{\psi}\| + \|\hat{\psi}\|_2)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $2 \leq q \leq \frac{2d}{2-d}$,
   (b) $\forall a > 0 \exists b > 0 : \|\hat{\psi}\|_p \leq a\||\eta|\hat{\psi}\| + b\|\hat{\psi}\|_2$,
   (c) potential $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ is infinitesimally form bounded with respect to he form $\|\nabla\psi\|^2$ for $p > \frac{d}{2}$.