Mathematical Methods of Quantum Mechanics
Exercise 2

Problems

Exercise 1 (Canonical Commutation Relations cannot hold for Bounded Operators).

Let $\mathscr{H}$ be a complex Hilbert space with $\dim\mathscr{H} \geq 1$. Let P, Q be two self-adjoint linear operators on $\mathscr{H}$ such that

$$\langle \varphi, [P, Q]\varphi \rangle := \langle P\varphi, Q\varphi \rangle - \langle Q\varphi, P\varphi \rangle = \frac{1}{\mathrm{i}} \|\varphi\|^2$$

for all $\varphi \in \mathscr{D}(P) \cap \mathscr{D}(Q)$.

(a) Show that $\dim\mathscr{H} = \infty$.
HINT: In a finite-dimensional Hilbert space, linear operators can be represented by matrices. Use the *trace*

$$\mathrm{tr}A = \sum_{j=1}^{\dim\mathscr{H}} \langle \psi_j, A\psi_j \rangle, \quad \{\psi_j\}_{j=1,\ldots,\dim\mathscr{H}} \text{ ONB of } \mathscr{H}.$$

(b) Assume that P and Q are bounded. Show that neither P nor Q can have have an eigenvalue.

(c) Show that neither P nor Q can be bounded.
HINT[1]: Every self-adjoint operator A has non-empty spectrum $\sigma(A)$. If $\lambda \in \sigma(A)$, there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset \mathscr{D}(A)$ such that $\|\phi_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(A - \lambda)\phi_n\| \to 0$ as $n \to \infty$.

Exercise 2 (Further Properties of Closed Operators, I).

Let $A : \mathscr{D}(A) \to \mathscr{H}$ be closed and symmetric. Prove that

(a) $\mathrm{Ran}(A - z_0) = \mathscr{H}$ for some $z_0 \in \mathbb{C}_\pm$ *if and only if* $\mathrm{Ran}(A - z) = \mathscr{H}$ for all $z \in \mathbb{C}_\pm$.

(b) $\dim(\mathrm{Ker}(A^* - z_0)) = d_\pm$ for some $z_0 \in \mathbb{C}_\pm$ *if and only if* $\dim(\mathrm{Ker}(A^* - z)) = d_\pm$ for all $z \in \mathbb{C}_\pm$.

Exercise 3 (Further Properties of Closed Operators, II).

Show that if A is closed and B is bounded, then AB is closed. Moreover, if B is injective and B^{-1} is bounded, then BA is closed.

[1]You can use this property without proof, it will be proved in a future exercise or in class.

Exercise 4 (Spectrum of Multiplication Operators). Let[2] $M_V : \mathcal{D}(M_V) \to L^2(\mathbb{R}^d, \mathrm{d}\mu)$ be the operator of multiplication with the measurable function $V : \mathbb{R}^d \to \mathbb{C}$,

$$(M_V\psi)(x) := V(x)\psi(x), \quad x \in \mathbb{R}^d.$$

Prove that M_V is closed on its maximal domain

$$\mathcal{D}(M_V) := \left\{\psi \in L^2(\mathbb{R}^d, \mathrm{d}\mu) : V\psi \in L^2(\mathbb{R}^d, \mathrm{d}\mu)\right\},$$

and show that the spectrum of M_V is given by the *essential range* of V, i.e.

$$\sigma(M_V) = \operatorname{ess\,ran}(V) := \left\{z \in \mathbb{C} \mid \forall \epsilon > 0 : \mu(\{x \in \mathbb{R}^d : |V(x) - z| < \epsilon\}) > 0\right\}.$$

What can you say about eigenvalues of M_V?

HINT: Calculate the resolvent set $\rho(M_V)$.

[2]If you are uncomfortable with abstract measure theory, you may think of μ being the d-dimensional Lebesgue measure.