Mathematical Methods of Quantum Mechanics

Exercise 2

PROBLEMS

Exercise 1 (Canonical Commutation Relations cannot hold for Bounded Operators).

Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 1$. Let $P, Q$ be two self-adjoint linear operators on $\mathcal{H}$ such that

$$\langle \varphi, [P, Q] \varphi \rangle := \langle P \varphi, Q \varphi \rangle - \langle Q \varphi, P \varphi \rangle = \frac{1}{i} \| \varphi \|^2$$

for all $\varphi \in \mathcal{D}(P) \cap \mathcal{D}(Q)$.

(a) Show that $\dim \mathcal{H} = \infty$.

HINT: In a finite-dimensional Hilbert space, linear operators can be represented by matrices. Use the trace

$$\text{tr} A = \sum_{j=1}^{\dim \mathcal{H}} \langle \psi_j, A \psi_j \rangle, \quad \{ \psi_j \}_{j=1,...,\dim \mathcal{H}} \text{ ONB of } \mathcal{H}.$$

(b) Assume that $P$ and $Q$ are bounded. Show that neither $P$ nor $Q$ can have an eigenvalue.

(c) Show that neither $P$ nor $Q$ can be bounded.

HINT: Every self-adjoint operator $A$ has non-empty spectrum $\sigma(A)$. If $\lambda \in \sigma(A)$, there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A)$ such that $\| \phi_n \| = 1$ for all $n \in \mathbb{N}$ and $\| (A - \lambda) \phi_n \| \to 0$ as $n \to \infty$.

Exercise 2 (Further Properties of Closed Operators, I).

Let $A : \mathcal{D}(A) \to \mathcal{H}$ be closed and symmetric. Prove that

(a) $\text{Ran}(A - z_0) = \mathcal{H}$ for some $z_0 \in \mathbb{C}_\pm$ if and only if $\text{Ran}(A - z) = \mathcal{H}$ for all $z \in \mathbb{C}_\pm$.

(b) $\dim (\text{Ker}(A^* - z_0)) = d_\pm$ for some $z_0 \in \mathbb{C}_\pm$ if and only if $\dim (\text{Ker}(A^* - z)) = d_\pm$ for all $z \in \mathbb{C}_\pm$.

Exercise 3 (Further Properties of Closed Operators, II).

Show that if $A$ is closed and $B$ is bounded, then $AB$ is closed. Moreover, if $B$ is injective and $B^{-1}$ is bounded, then $BA$ is closed.

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1You can use this property without proof, it will be proved in a future exercise or in class.
Exercise 4 (Spectrum of Multiplication Operators). Let $M_V : \mathcal{D}(M_V) \to L^2(\mathbb{R}^d, d\mu)$ be the operator of multiplication with the measurable function $V : \mathbb{R}^d \to \mathbb{C}$, 
\[
(M_V \psi)(x) := V(x)\psi(x), \quad x \in \mathbb{R}^d.
\]
Prove that $M_V$ is closed on its maximal domain 
\[
\mathcal{D}(M_V) := \left\{ \psi \in L^2(\mathbb{R}^d, d\mu) : V \psi \in L^2(\mathbb{R}^d, d\mu) \right\},
\]
and show that the spectrum of $M_V$ is given by the essential range of $V$, i.e. 
\[
\sigma(M_V) = \text{ess ran}(V) := \left\{ z \in \mathbb{C} \mid \forall \epsilon > 0 : \mu(\{ x \in \mathbb{R}^d : |V(x) - z| < \epsilon \}) > 0 \right\}.
\]
What can you say about eigenvalues of $M_V$?

HINT: Calculate the resolvent set $\rho(M_V)$.

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2If you are uncomfortable with abstract measure theory, you may think of $\mu$ being the $d$-dimensional Lebesgue measure.