Mathematical Methods of Quantum Mechanics
Exercise 3

Problems

1. Let T_0 be a symmetric operator acting as a second derivative. Find all self-adjoint extensions of the operator T_0 for the following domains
   (a) $\mathcal{D}(T_0) = \{f \in AC^2(\mathbb{R}^+) | f(0) = f'(0) = 0\}$,
   (b) $\mathcal{D}(T_0) = \{f \in AC^2((0,1)) | f(0) = f'(0) = 0 = f(1) = f'(1)\}$.
2. Find an example of symmetric operator A with the following properties:
   (a) Operator is maximal and not self-adjoint.
   Hint: Try to think about momentum operator in various settings.
   (b) Operator has a symmetric extension and no self-adjoint extension.
   Hint: Try to modify the previous solution.

Definitions

- **Symmetric extension**
  Let A be a symmetric operator. Any symmetric A' such that $A \subset A'$ is called symmetric extension of operator A.
- **Maximal operator**
  A symmetric operator is maximal if it has no proper symmetric extensions A', i.e. the relation $A \subset A'$ implies $A = A'$.
- **Set of functions** $AC^2(I)$
  Let a function f be in $AC^2(I)$ then $f \in L^2(I)$, f is differentiable, f' is absolutely continuous and $f'' \in L^2(I)$.
- **Absolute continuity**
  The function $f : I \to \mathbb{C}$ is absolutely continuous if for given $\epsilon > 0$ exists $\delta > 0$ such that

$$\sum_j |f(x_j) - f(y_j)| < \epsilon \tag{1}$$

  holds for all possible finites choices of disjoint subintervals $(x_j - y_j) \subset I$ satisfying $\sum_j |x_j - y_j| < \delta$.

Extension cookbook in 7 steps

How to get a symmetric extension almost everytime and a self-adjoint one sometimes

1. Take an symmetric operator A
2. Make it adjoint operator A^*
3. Find deficiency subspaces $\mathcal{K}_\pm = \mathrm{Ker}(i \mp A^*) = \mathrm{Ran}(i \pm A)^\perp$ of A.
4. Check whether both deficiency indices $n_\pm = \dim(\mathcal{K}_\pm)$ are non-zero.
   (a) If at least one $n_\pm$ is 0 we already have maximal symmetric extension and there is nothing more one can do.
   (b) If both are $n_\pm = 0$, the operator A is already self-adjoint.
   (c) If both are $n_\pm \neq 0$ we can continue.
5. Choose appropriate partial isometry U from subspace of $\mathcal{K}_+$ into $\mathcal{K}_-$.
6. Define the new operator as

$$A_U(\phi + \phi_+ + U\phi_+) = A\phi + i\phi_- - iU\phi_+ \tag{2}$$

   with the domain

$$\mathcal{D}(A_U) = \{\phi + \phi_+ + U\phi_+ | \phi \in \mathcal{D}(A), \phi \in I(U)\} \tag{3}$$

   where space $I(U) \subset \mathcal{K}_+$ is such a space that U restricted to it is isometry.
7. We have a symmetric extension A_U of A.

Further Reading

1. M. Reed, B. Simon:*Fourier Analysis, Self-Adjointness (Methods of Modern Mathematical Physics, Vol. 2)*, Academic Press, 1975, p. 135-146
2. J. Blank, P. Exner, M. Havlíček: *Hilbert-Space Operators in Quantum Physics*, Springer Netherlands, 2008, p. 117-126