Problems

1. Let $V$ and $T$ be densely defined linear operators on a Hilbert space $\mathcal{H}$ with $\mathcal{D}(T) \subset \mathcal{D}(V)$.

Assume that:

(a) For some $a, b \in \mathbb{R}$ and all $\psi \in \mathcal{D}(T)$

$$\|V\psi\| \leq a\|T\psi\| + b\|\psi\|. \quad (1)$$

(b) For some $\tilde{a}, \tilde{b} \in \mathbb{R}$ and all $\psi \in \mathcal{D}(T)$

$$\|V\psi\|^2 \leq \tilde{a}^2\|T\psi\|^2 + \tilde{b}^2\|\psi\|^2. \quad (2)$$

Show the following statements:

(a) If (2) holds, then (1) holds with $a = \tilde{a}$ and $b = \tilde{b}$.

(b) If (1) holds, then (2) holds for $\tilde{a}^2 = (1 + \epsilon)a^2$ and $\tilde{b}^2 = (1 + \epsilon^{-1})b^2$ for each $\epsilon > 0$. Thus the infimum of all $a$ in (1) is the same as the infimum of all $\tilde{a}$ in (2).

2. In this exercise we want to prove by hand that $-\Delta + V$ is self-adjoint on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$ if $V \in L^2(\mathbb{R}^3)$ and real valued.

(a) By the functional calculus, we can represent the resolvent of $-\Delta$ as

$$(-\Delta + E)^{-1} = \int_0^\infty e^{-tE}e^{-t\Delta} dt, \quad E > 0.$$ 

Using the heat kernel

$$e^{-t\Delta}(x, y) = (4\pi t)^{-d/2}e^{-|x-y|^2/4t}$$

show that for $d \geq 3$ the kernel of $(-\Delta + E)^{-1}$ satisfies

$$0 \leq (-\Delta + E)^{-1}(x, y) \leq c_d|x - y|^{2-d}e^{-2\sqrt{E}|x-y|}.$$ 

HINT: Try to prove that

$$(-\Delta + E)^{-1}(x, y) = (4\pi)^{-d/2}|x - y|^{2-d}\int_0^\infty \exp\left(-|x - y|^2Es - \frac{1}{4s}\right)s^{-d/2} ds,$$

and use that $(a + a^{-1}) \geq 2$ for all $a > 0$.

(b) Let $V \in L^2(\mathbb{R}^3)$. Prove that the operator $V(-\Delta + E)^{-1}$ is bounded on $L^2(\mathbb{R}^3)$ with $\|V(-\Delta + E)^{-1}\| \to 0$ as $E \to \infty$.

HINT: Try to estimate $\|V(-\Delta + E)^{-1}\varphi\|_2$ for $\varphi \in L^2(\mathbb{R}^3)$. $(-\Delta + E)^{-1}$ is a convolution operator, use Cauchy-Schwarz for convolutions at the right place.

(c) Conclude that in three dimensions any $V \in L^2(\mathbb{R}^3)$ is a Kato potential, i.e. for any $a > 0$ there exists $b \geq 0$ such that

$$\|V\varphi\|_2 \leq a\|\Delta\varphi\|_2 + b\|\varphi\|_2, \quad \varphi \in H^2(\mathbb{R}^3).$$