Mathematical Methods of Quantum Mechanics
Exercise 4

Problems

1. Let V and T be densely defined linear operators on a Hilbert space $\mathcal{H}$ with $\mathcal{D}(T) \subset \mathcal{D}(V)$. Assume that:
   (a) For some $a, b \in \mathbb{R}$ and all $\psi \in \mathcal{D}(T)$
   $$\|V\psi\| \leq a\|T\psi\| + b\|\psi\|\,. \tag{1}$$
   (b) For some $\tilde{a}, \tilde{b} \in \mathbb{R}$ and all $\psi \in \mathcal{D}(T)$
   $$\|V\psi\|^2 \leq \tilde{a}^2\|T\psi\|^2 + \tilde{b}^2\|\psi\|^2\,. \tag{2}$$

   Show the following statements:
   (a) If (2) holds, then (1) holds with $a = \tilde{a}$ and $b = \tilde{b}$.
   (b) If (1) holds, then (2) holds for $\tilde{a}^2 = (1+\epsilon)a^2$ and $\tilde{b}^2 = (1+\epsilon^{-1})b^2$ for each $\epsilon > 0$. Thus the infimum of all a in (1) is the same as the infimum of all $\tilde{a}$ in (2).
2. In this exercise we want to prove by hand that $-\Delta + V$ is self-adjoint on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$ if $V \in L^2(\mathbb{R}^3)$ and real valued.
   (a) By the functional calculus, we can represent the resolvent of $-\Delta$ as
   $$(-\Delta + E)^{-1} = \int_0^\infty \mathrm{e}^{-tE}\mathrm{e}^{-t\Delta}\,\mathrm{d}t, \quad E > 0.$$
   Using the *heat kernel*
   $$\mathrm{e}^{-t\Delta}(x,y) = (4\pi t)^{-d/2}\mathrm{e}^{-\frac{|x-y|^2}{4t}}$$
   show that for $d \geq 3$ the kernel of $(-\Delta+E)^{-1}$ satisfies
   $$0 \leq (-\Delta + E)^{-1}(x,y) \leq c_d|x-y|^{2-d}\mathrm{e}^{-2\sqrt{E}|x-y|}.$$
   HINT: Try to prove that
   $$(-\Delta + E)^{-1}(x,y) = (4\pi)^{-d/2}|x-y|^{2-d}\int_0^\infty \exp\left(-|x-y|^2Es - \frac{1}{4s}\right)s^{-d/2}\,\mathrm{d}s,$$
   and use that $(a + a^{-1}) \geq 2$ for all $a > 0$.
   (b) Let $V \in L^2(\mathbb{R}^3)$. Prove that the operator $V(-\Delta+E)^{-1}$ is bounded on $L^2(\mathbb{R}^3)$ with $\|V(-\Delta+E)^{-1}\| \to 0$ as $E \to \infty$.
   HINT: Try to estimate $\|V(-\Delta+E)^{-1}\varphi\|_2$ for $\varphi \in L^2(\mathbb{R}^3)$. $(-\Delta+E)^{-1}$ is a *convolution operator*, use Cauchy-Schwarz for convolutions at the right place.
   (c) Conclude that in three dimensions any $V \in L^2(\mathbb{R}^3)$ is a Kato potential, i.e. for any $a > 0$ there exists $b \geq 0$ such that
   $$\|V\varphi\|_2 \leq a\|\Delta\varphi\|_2 + b\|\varphi\|_2, \quad \phi \in H^2(\mathbb{R}^3).$$