Mathematical Methods of Quantum Mechanics
Exercise 5

Problems

1. Let $A \in \mathcal{L}(H)$ and A be symmetric. Show that the operator-valued function on $\mathbb{R}$
$$U(t) := e^{-itA} = \sum_{n=0}^{\infty} \frac{(-itA)^n}{n!}$$
exists, is bounded in the operator norm and satisfies the following:
   (a) $\lim_{t\to 0} \left\| e^{-itA} - \mathbb{I} \right\| = 0$,
   (b) $\lim_{t\to 0} \left\| \frac{e^{-itA}-\mathbb{I}}{t} + iA \right\| = 0$,
   (c) $U(t)U(s) = U(t+s)$ for all $t, s \in \mathbb{R}$,
   (d) $\lim_{h\to 0} \|U(t+h) - U(t)\| = 0$ for all $t \in \mathbb{R}$,
   (e) $\lim_{h\to 0} \frac{U(t+h)-U(t)}{h} = -iAU(t) = -iU(t)A$ for all $t \in \mathbb{R}$,
   (f) $[U(t)]^* = U(-t)$ for all $t \in \mathbb{R}$.
2. Let $A, B \in \mathcal{L}(H)$. Prove the Trotter product formula
$$e^{A+B} = \lim_{N\to\infty} \left(e^{A/N} e^{B/N}\right)^N .$$
Hint: Use notation $C_N = e^{(A+B)/N}$ and $D_N = e^{A/N}e^{B/N}$ to show that $\|C_N + D_N\| = \mathcal{O}(N^{-2})$ and compare telescopic sums C_N^N and D_N^N.