Mathematical Methods of Quantum Mechanics  
Exercise 5  

Problems

1. Let \( A \in \mathcal{L}(H) \) and \( A \) be symmetric. Show that the operator-valued function on \( \mathbb{R} \)

\[
U(t) := e^{-itA} = \sum_{n=0}^{\infty} \frac{(-itA)^n}{n!}
\]

exists, is bounded in the operator norm and satisfies the following:

(a) \( \lim_{t \to 0} \| e^{-itA} - I \| = 0 \),

(b) \( \lim_{t \to 0} \left\| \frac{e^{-itA} - I}{t} + iA \right\| = 0 \),

(c) \( U(t)U(s) = U(t+s) \) for all \( t, s \in \mathbb{R} \),

(d) \( \lim_{h \to 0} \| U(t+h) - U(t) \| = 0 \) for all \( t \in \mathbb{R} \),

(e) \( \lim_{h \to 0} \frac{U(t+h)-U(t)}{h} = -iAU(t) = -iU(t)A \) for all \( t \in \mathbb{R} \),

(f) \( [U(t)]^* = U(-t) \) for all \( t \in \mathbb{R} \).

2. Let \( A, B \in \mathcal{L}(H) \). Prove the Trotter product formula

\[
e^{A+B} = \lim_{N \to \infty} \left( e^{A/N} e^{B/N} \right)^N.
\]

Hint: Use notation \( C_N = e^{(A+B)/N} \) and \( D_N = e^{A/N} e^{B/N} \) to show that \( \| C_N + D_N \| = \mathcal{O}(N^{-2}) \) and compare telescopic sums \( C_N^N \) and \( D_N^N \).