Mathematical Methods of Quantum Mechanics
Exercise 6

Problems

1. For every $t \in \mathbb{R}$ we define the bounded linear operator $e^{i\Delta t} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$e^{i\Delta t} = \mathcal{F}^{-1} \circ M_{V(k)} \circ \mathcal{F}\,,$$

where $\mathcal{F}$ is the Fourier transform operator on $L^2(\mathbb{R}^n)$ and $M_{V(k)}$ is the multiplication operator by $V(k) = \exp(-i|k|^2 t)$. Show the following

(a) $\forall \psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$:

$$\left(e^{i\Delta t}\psi\right)(x) = exp\left(-i\frac{\pi}{4}n\right)\int_{\mathbb{R}^n} \frac{\exp\left(i\frac{|x-y|^2}{4t}\right)}{(4\pi t)^{n/2}}\psi(y)dy$$

a.e. in $\mathbb{R}^n$.

(b) $\forall \psi \in L^2(\mathbb{R}^n)$:

$$\lim_{\epsilon\to 0}\left\| e^{i\Delta t}\psi - e^{-i\frac{\pi}{4}n}\int_{\mathbb{R}^n} \frac{\exp\left(i\frac{|x-y|^2}{4t} - \epsilon|y|^2\right)}{(4\pi t)^{n/2}}\psi(y)dy\right\|_{L^2(\mathbb{R}^n)} = 0\,.$$

(c) $\forall \psi \in S(\mathbb{R}^n)$, $\phi(t,x) := \left(e^{i\Delta t}\psi\right)(x)$ satisfies

$$i\partial_t \phi(t,x) = -\Delta_x \phi(t,x)$$

in $C^\infty((\mathbb{R}\setminus\{0\}) \times \mathbb{R}^n)$.

2. Let $\lambda > 0$ then the operator $U_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined as

$$[U_\lambda \phi](x) := \lambda^{n/2}\phi(\lambda x)\,.$$

Show that:

(a) U_λ is unitary and $U_\lambda^{-1} = U_{\lambda^{-1}}$,

(b) U_λ leaves $H^s(\mathbb{R}^n)$ invariant,

(c) On $\mathcal{S}(\mathbb{R}^n)$ the following holds

$$U_\lambda \Delta U_\lambda^{-1} = \frac{1}{\lambda^2}\Delta\,, \quad U_\lambda V(x) U_\lambda^{-1} = V(\lambda x)\,,$$

with V being multiplication operator satisfying $V\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$.

3. Consider operators $H_0 = -\frac{\hbar^2}{2m}\Delta + Cr^k$ and $H_1 = -\Delta + r^k$ acting on $\mathcal{S}(\mathbb{R}^n)$. Let the operator H_0 have an eigenfunction ψ_0, $\|\psi_0\| = 1$ with the eigenvalue E_0, i.e.

$$H_0\psi_0 = E_0\psi_0\,.$$

Show that the operator H_1 has an eigenfunction ψ_1 with an eigenvalue E_1. Furthermore derive the relation between ψ_1, E_1 and ψ_0, E_0.
Hint: Use the solution of previous Problem.