Mathematical Methods of Quantum Mechanics
Exercise 7

Problems

1. Prove the following statement: Let H be a self-adjoint operator on $L^2(\mathbb{R}^n)$ and let function $f \in \mathbb{L}^\infty(\mathbb{R}^n)$ satisfy $\lim_{|x|\to\infty} f(x) = 0$. Then the following are equivalent:
   (a) $\exists z_0 \in \rho(H) : F(H - z_0)^{-1}$ is compact,
   (b) $\forall z \in \rho(H) : F(H - z)^{-1}$ is compact

   where F is multiplication operator by a function $f(x)$.
2. Let $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $f(x) = g(x)|x|_\epsilon^{-\alpha} = \frac{g(x)}{(\epsilon+|x|^2)^{\alpha/2}}$ with $d \geq 3$. Then the following holds
   (a) $\int_{\mathbb{R}^n} |\nabla f|^2 dx = \int_{\mathbb{R}^n} |\nabla g|^2 |x|_\epsilon^{-2} dx + \int_{\mathbb{R}^n} |f|^2 |x|_\epsilon^{-2} [\alpha n - (\alpha^2 + 2\alpha)|x|_\epsilon^{-2} x^2] dx$,
   (b) $\int_{\mathbb{R}^n} |\nabla f|^2 dx \geq (-\alpha^2 + (d-2)\alpha) \int_{\mathbb{R}^n} |f|^2 |x|^{-2} dx$.

   Furthermore find an optimal value of *alpha*.
3. *Hardy's inequality for 1D* Show that for every function $\psi \in H_0^1(\mathbb{R}^+)$ the following holds
$$\int_{\mathbb{R}+} |\psi'(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}+} \frac{|\psi(x)|^2}{x^2} dx \,. \tag{1}$$
4. Show that the inequality (1) is non-attainable and optimal, i.e.
   (a) *non-attainability* There is no non-trivial function such that, the equality holds in (1),
   (b) *optimality* We can not make the constant 1/4 on the right hand side bigger.