Mathematical Methods of Quantum Mechanics
Exercise 8

Problems

1. Show that the operator $V(-\Delta + i)^{-1}$ is compact if:
   (a) $V \in L^2(\mathbb{R}^d)$ with $d \leq 3$,
   (b) $V \in L^p(\mathbb{R}^n) + L^\infty_\epsilon(\mathbb{R}^n)$ with $p = 2$ if $n = 1, 2, 3$ and $p > n/2$ if $n \geq 4$.

   Remark: Let A be a closed operator and V satisfy $\mathcal{D}(V) \subseteq \mathcal{D}(A)$. Then the following are equivalent:
   (a) $\exists z_0 \in \rho(H) : V(A - z_0)^{-1}$ is compact,
   (b) $\forall z \in \rho(H) : V(A - z)^{-1}$ is compact.

2. Let
$$H = -\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2 + V(x_1 - x_2)$$
   be a two particle Hamiltonian on $H^2(\mathbb{R}^6)$ where $-\frac{1}{2m_i}\Delta_i$ is the kinetic energy and x_i is a position of i-th particle respectively. Using the center of the mass coordinates
$$y_{CM} = \frac{m_1 x_1}{M} + \frac{m_2 x_2}{M}\,, \quad y = x_1 - x_2$$
   where $M = m_1 + m_2$, we can obtain the following unitarily equivalent operator
$$\tilde{H} = -\frac{1}{2M}p_{CM}^2 - \frac{1}{2\mu}\Delta_2 + V(y)$$
   where $\mu = \frac{M}{m_1 m_2}$ with the domain $H^2(\mathbb{R}^6, Cdx)$ where C is appropriate constant. Furthermore calculate the constant C.

3. We define the following operator valued function
$$U(\phi)\psi(x) = \psi(R(-\phi)x)\,, \quad R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$
   where $R(\phi)$ corresponds to the rotation of coordinates. Show that
   (a) $U(t)$ is a strongly continuous one-parameter unitary group,
   (b) the operator $L = -i(x_1\partial_{x_2} - x_2\partial_{x_1})$ is the generator of $U(t)$.