Mathematical Methods of Quantum Mechanics
Exercise 9

Problems

1. Let the operator $H_0 = P^2$ be defined on $\mathcal{H}^2(\mathbb{R}^d)$ with $P = -i\nabla$. We define the following cut-off function $\xi_R(x) = \xi(x/R)$ with $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ satisfying $\xi(x) = 1$ for $0 \leq |x| \leq 1$ and $\operatorname{supp} \xi \in B_2(0)$. Furthermore we introduce multiplication operator V which is relatively H_0-bounded with relative bound $a < 1$. Show that the following holds:
   (a) $\lim_{R\to\infty} \|[H_0, \xi_R](H_0 + 1)^{-1}\| = 0\,,$
   (b) $\lim_{R\to\infty} \|[H_0 + V, \xi_R](H_0 + V + i)^{-1}\| = 0\,.$
2. Let H be a self-adjoint operator bounded from below satisfying
$$\lim_{R\to\infty} \|[H, \xi_R](H + i)^{-1}\| = 0\,.$$
   We introduce the following quantity
$$\Sigma_R(H) = \inf\{(\phi, H\phi)|\phi \in \mathcal{D}(H), \|\phi\| = 1, \operatorname{supp}\phi \in B_R(0)^C\}$$
   and the notation $\alpha = \lim_{R\to\infty} \Sigma_R(H)$ along with $\lambda_0 = \inf \sigma_{ess}(H)$. Show the following:
   (a) $\forall R < \infty : \lambda_0 \geq \Sigma_R(H)$,
   (b) $\exists \phi_n \in \mathcal{D}(H), \|\phi_n\| = 1, \operatorname{supp}\phi_n \subset B_{R_n}(0)^C, R_n \to \infty$ such that $\lim_{n\to\infty}(\phi_n, H\phi_n) = \alpha$,
   (c) $\lim_{n\to\infty} \|P_\epsilon \phi_n\| = 0$ where for $\epsilon > 0$ we denote the spectral projector of the operator H as $P_\epsilon = P_H(-\infty, \lambda_0 - \epsilon)$,
   (d) $(\phi, HP_\epsilon^\perp \phi) \geq (\lambda_0 - \epsilon)\|P_\epsilon^\perp \phi\|^2$ where $P_\epsilon + P_\epsilon^\perp = \mathbb{I}$,
   (e) $\alpha \geq \lambda_0$,
   (f) $\alpha = \lambda_0$.