MMQM Exercise 1

Problem 1: Sesquilinear and quadratic forms

$\mathcal{D}$ vector space, $s: \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ sesquilinear form,

i.e. $s(x, \alpha y + \beta z) = \alpha s(x,y) + \beta s(x,z)$

linear in second component $\quad \forall x,y,z \in \mathcal{D}$, $\alpha, \beta \in \mathbb{C}$

$s(\alpha x + \beta y, z) = \bar{\alpha}\, s(x,z) + \bar{\beta}\, s(y,z)$

conjugate linear in ~~second~~ first component

Associated quadratic form $q: \mathcal{D} \to \mathbb{C}$, $q(x) = s(x,x)$.

Typical example: ① $A: \mathcal{D}(A) \to \mathfrak{H}$ linear operator on Hilbert space $\mathfrak{H}$. Then $s_A: \mathcal{D}(A) \times \mathcal{D}(A) \to \mathbb{C}$,

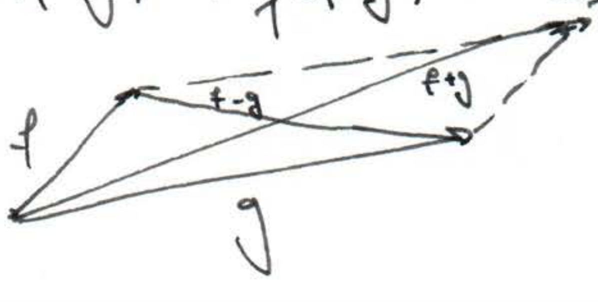
$$s_A(\varphi, \psi) := \langle \varphi, A\psi \rangle$$

is sesquilinear form on $\mathfrak{H}$ with corresponding quadratic form $q_A: \mathcal{D}(A) \to \mathbb{C}$, $q_A(\psi) = \langle \psi, A\psi \rangle$.

② Inner product $\langle \cdot, \cdot \rangle$ in (complex) Hilbert space $\mathfrak{H}$ with corresponding quadratic form $q(\psi) = \|\psi\|^2$.

(a) q satisfies parallelogram law

$$q(f+g) + q(f-g) = 2q(f) + 2q(g) \quad \forall f,g \in \mathcal{D}.$$



Proof (by calculation):

$$q(f+g) + q(f-g) = s(f+g, f+g) + s(f-g, f-g)$$

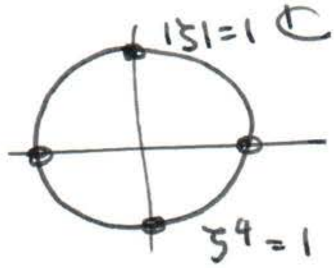
$$= s(f,f) + s(g,g) + s(f,g) + s(g,f)$$
$$+ s(f,f) + s(-g,-g) \mp s(f, \pm g) \mp s(\pm g, f)$$

$$= 2q(f) + 2q(g).$$ ■

(b) Polarisation Identity

$$s(f,g) = \frac{1}{4}\left[q(f+g) - q(f-g) \right] + \frac{i}{4}\left[q(f-ig) - q(f+ig) \right]$$

$$\overset{(*)}{=} \frac{1}{4} \sum_{\zeta^4 = 1} \zeta \, q(\zeta f + g)$$



Proof (by calculation):

Let $\zeta \in \mathbb{C}$, $|\zeta| = 1$. Then

$$\zeta q(\zeta f + g) = \zeta \, s(\zeta f + g, \zeta f + g)$$
$$= \zeta \left(|\zeta|^2 s(f,f) + s(g,g) + \bar{\zeta} s(f,g) + \zeta s(g,f) \right)$$
$$= \zeta q(f) + \zeta q(g) + |\zeta|^2 s(f,g) + \zeta^2 s(g,f)$$
$$= \zeta (q(f) + q(g)) + s(f,g) + \zeta^2 s(g,f).$$

Since $\sum_{\zeta^4=1} \zeta = 0$, $\sum_{\zeta^4=1} \zeta^2 = 0$, we have

$$\frac{1}{4} \sum_{\zeta^4=1} \zeta q(\zeta f + g) = s(f,g).$$ ■

(c) s symmetric $\iff$ q real-valued

$\left(s(f,g) = \overline{s(g,f)} \ \forall f,g \in \mathcal{D} \right)$ $\quad$ $\left(q(f) \in \mathbb{R} \ \forall f \in \mathcal{D} \right)$

Proof: "$\Rightarrow$" If s is symmetric, then

$$q(f) = s(f,f) = \overline{s(f,f)} = \overline{q(f)} \quad \forall f \in \mathcal{D} \Rightarrow q(f) \in \mathbb{R}.$$

"$\Leftarrow$" Assume $q(f) \in \mathbb{R}$ $\forall f \in \mathcal{D}$. Then by polarisation

$$s(f,g) = \frac{1}{4} \sum_{\zeta^4=1} \zeta\, q(\zeta f + g) \underset{\overline{\{\zeta^4=1\}} = \{\zeta^4=1\}}{=} \frac{1}{4} \sum_{\zeta^4=1} \bar{\zeta}\, q(\bar{\zeta} f + g)$$

$$\overset{q \in \mathbb{R}}{=} \frac{1}{4} \sum_{\zeta^4=1} \overline{\zeta q(\bar{\zeta} f + g)} \underset{\circledast}{=} \frac{1}{4} \sum_{\zeta^4=1} \overline{\zeta q(f + \zeta g)}$$

$$= \overline{s(g,f)}.$$

$\circledast$ $|\zeta| = 1$, $q(\bar{\zeta} f + g) = s(\bar{\zeta} f + g, \bar{\zeta} f + g)$
$= |\zeta|^2 s(\bar{\zeta} f + g, \bar{\zeta} f + g)$
$= \zeta\bar{\zeta}\, s(\bar{\zeta} f + g, \bar{\zeta} f + g)$
$= s(|\zeta|^2 f + \zeta g, |\zeta|^2 f + \zeta g)$
$= s(f + \zeta g, f + \zeta g) =$
$= q(f + \zeta g).$

$\blacksquare$

Problem 2: operator norm of inverse

$A \in \mathcal{L}(X)$ bijective, i.e. $A : X \to X$ bounded, linear, injective, surjective

$\ker A = \{0\}$ $\quad$ $\operatorname{Ran} A = X$

Then
$$\|A^{-1}\|^{-1} = \inf_{f \in X, \|f\|=1} \|Af\| .$$

Proof. Recall that $\|A\| = \sup\limits_{f \neq 0} \dfrac{\|Af\|}{\|f\|} = \sup\limits_{\|f\|=1} \|Af\|$.

Then
$$\|A^{-1}\|^{-1} = \left(\sup_{f \neq 0} \frac{\|A^{-1}f\|}{\|f\|} \right)^{-1} = \inf_{f \neq 0} \frac{\|f\|}{\underbrace{\|A^{-1}f\|}_{\neq 0 \text{ since } A^{-1} \text{ inj.}}}$$
$$= \inf_{g \neq 0} \frac{\|Ag\|}{\|g\|} = \inf_{\|g\|=1} \|Ag\| .$$

since by bijectivity of A, $\forall f \in X \; \exists! \, g \in X : f = Ag$,

i.e. $X \setminus \{0\} = \operatorname{Ran} A \setminus \{0\} = \{ f \in X \setminus \{0\} : \exists g \in X : f = Ag \}$

∎

Problem 3: $\mathrm{Inv}(X)$ is open

$$\mathrm{Inv}(X) = \{ A \in \mathcal{B}(X) : A \text{ bijective and } A^{-1} \text{ bounded} \}.$$

Then $\mathrm{Inv}(X)$ is open in $\mathcal{B}(X)$, i.e. for each $A \in \mathrm{Inv}(X)$ there is a neighbourhood $U_r(A) \subset \mathrm{Inv}(X)$.

Proof: We will use the Neumann series:

Let $T \in \mathcal{B}(X)$. If $\sum_{k=0}^{\infty} T^k$ converges in operator norm

(for instance if $\|T\| < 1$, since then

$$\lim_{M,N\to\infty} \Big\| \sum_{k=0}^{M} T^k - \sum_{k=0}^{N} T^k \Big\| = \lim_{M,N\to\infty} \Big\| \sum_{k=M+1}^{N} T^k \Big\|$$

$$\overset{\Delta\text{-ineq.}, \|AB\| \le \|A\|\|B\|}{\le} \lim_{M,N\to\infty} \sum_{k=M+1}^{N} \|T\|^k = 0 \quad \text{since} \quad \sum_{k=0}^{\infty} \|T\|^k = \frac{1}{1-\|T\|} \text{ if } \|T\| < 1$$

So $\sum_{k=0}^{M} T^k$ is Cauchy sequence in $\mathcal{B}(X)$ which has a limit by completeness of $\mathcal{B}(X)$.)

then $\mathbb{1} - T$ is invertible and

$$(\mathbb{1} - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Now let $A \in \mathrm{Inv}(X)$, $B \in \mathcal{B}(X)$.

Then $B = A + B - A = A - (A - B) = A(\mathbb{1} - A^{-1}(A-B))$.

Neumann series: If $\|A^{-1}(A-B)\| < 1$, then

$\mathbb{1} - A^{-1}(A-B)$ invertible with

$$\left(\mathbb{1} - A^{-1}(A-B)\right)^{-1} = \sum_{k=0}^{\infty} \left[A^{-1}(A-B)\right]^k$$

and $$\left\|\left(\mathbb{1} - A^{-1}(A-B)\right)^{-1}\right\| \leq \sum_{k=0}^{\infty} \|A^{-1}(A-B)\|^k = \frac{1}{1-\|A^{-1}(A-B)\|}$$

$$\leq \frac{1}{1-\|A^{-1}\|\|A-B\|}.$$

In particular: if $\|A-B\| < \frac{1}{\|A^{-1}\|}$, then

$\|A^{-1}(A-B)\| \leq \|A^{-1}\|\|A-B\| < 1$ and thus

$\mathbb{1} - A^{-1}(A-B)$ is inv. with bounded inverse,

so $\mathbb{1} - A^{-1}(A-B) \in \mathrm{Inv}(X)$.

$\Rightarrow$ $B = A\left(\mathbb{1} - A^{-1}(A-B)\right)$ invertible with bdd inv.

$$B^{-1} = \left(\mathbb{1} - A^{-1}(A-B)\right)^{-1} A^{-1},$$

$$\|B^{-1}\| \leq \left\|\left(\mathbb{1} - A^{-1}(A-B)\right)^{-1}\right\|\|A^{-1}\|$$

$$\leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\|A-B\|}, \text{ so } B \in \mathrm{Inv}(X).$$

This means that if $A \in \mathrm{Inv}(X)$, then

$B(A, \|A^{-1}\|^{-1}) \subset \mathrm{Inv}(X)$, where $B(A,r) = \{B \in \mathcal{L}(X) : \|A-B\| < r\}$.

In particular $\mathrm{Inv}(X)$ open. ∎

Problem 4: Closable Operators

Problem with unbounded operators:

if $\|Az_n\| \xrightarrow{n\to\infty} \infty$ along a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, $\|z_n\|=1$,

then essentially two phenomena occur:

(1) loss of continuity (linear bdd. $\Leftrightarrow$ linear cont.)

It may happen that $\underset{\mathcal{D}(A)}{\overset{\cap}{\tilde{x}_n, x_n}} \longrightarrow x \in \mathcal{H}$ but

- Ax_n has no limit
- Ax_n, $A\tilde{x}_n$ have different limits

Moreover, if $x \in \mathcal{D}(A)$, it could be that $Ax_n \to y \neq Ax$.

(2) loss of bounded linear extension

cannot extend A by continuity from $\mathcal{D}(A)$

to lin. operator on $\overline{\mathcal{D}(A)}$ ($=\mathcal{H}$ if A densely defined)

(no bounded linear extension theorem)

Sometimes, the situation is not so bad:

For all sequences $\underset{\mathcal{D}(A)}{\overset{\cap}{x_n}} \longrightarrow x \in \mathcal{H}$ along which Ax_n has a limit, this limit is unique!

In this case, we can extend the operator.

Natural extension: closure of A

$$\mathcal{D}(\overline{A}) = \{x \in \mathcal{H} : \exists y \in \mathcal{H} \text{ s.t. for any sequence } \mathcal{D}(A) \ni x_n \to x, \; Ax_n \to y\} \supseteq \mathcal{D}(A)$$

$\bar{A}x := y$. (y uniquely identified by x!)

This extension is linear (check!) and is closed, i.e.

if $x \in \mathcal{H}$ is a limit point of $\mathcal{D}(A)$ such that

$\mathcal{D}(A) \ni x_n \to x$ and $Ax_n \to y$ for some $y \in \mathcal{H}$, then

$x \in \mathcal{D}(A)$ and $Ax = y$.

$\Rightarrow$ A is closable.

The notion of closability is intimately related to the graph $\Gamma(A)$ of A,

$$\Gamma(A) = \{(x, Ax) \in \mathcal{H}^2 : x \in \mathcal{D}(A)\}$$

naturally equipped with direct sum topology, making it Hilbert space with

$$\langle (z,w), (\tilde{z},\tilde{w}) \rangle_{\mathcal{H}\oplus\mathcal{H}} := \langle z, \tilde{z} \rangle + \langle w, \tilde{w} \rangle .$$

If $\overline{\Gamma(A)}$ denotes the closure of $\Gamma(A)$ in $\mathcal{H}\oplus\mathcal{H}$, then A is closed iff $\overline{\Gamma(A)} = \Gamma(A)$ and

A is closable iff $\overline{\Gamma(A)} = \Gamma(B)$ for some linear operator B on $\mathcal{H}$.

(Indeed $B = \bar{A}$ is the canonical (since minimal) extension s.t. $\overline{\Gamma(A)} = \Gamma(\bar{A})$.

Conversely, if $\overline{\Gamma(A)} = \Gamma(B)$ for some B, then B is a closed extension of A)

For completeness:

- A closed $\Leftrightarrow$ $\overline{\Gamma(A)} = \Gamma(A)$.

Let A be closed. Since $\Gamma(A) \subseteq \overline{\Gamma(A)}$, we only need to check that every $(x,y) \in \overline{\Gamma(A)}$ also lies in $\Gamma(A)$.

So let $(x,y) \in \overline{\Gamma(A)}$. Then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ s.t. $(x_n, Ax_n) \xrightarrow{n\to\infty} (x,y)$ in $\mathcal{H}\oplus\mathcal{H}$, in particular $x_n \to x$ and $Ax_n \to y$. Since A is closed, we have $x \in \mathcal{D}(A)$ and $Ax = y$, which means $(x, Ax) = (x,y) \in \Gamma(A)$.

Conversely, let $\overline{\Gamma(A)} = \Gamma(A)$. Then for every $(x, Ax) \in \Gamma(A)$ there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ s.t. $x_n \to x \in \mathcal{D}(A)$ and $Ax_n \to Ax$, since

$(x_n, Ax_n) \to (x, Ax)$ in $\mathcal{H}\oplus\mathcal{H}$.

But this means A is closed. ∎

- A closable $\Leftrightarrow$ $\overline{\Gamma(A)} = \Gamma(B)$ for some lin. op. B on $\mathcal{H}$

Let A be closable. Then A has a closed extension B s.t. $\mathcal{D}(B) \supseteq \mathcal{D}(A)$ and $Ax = Bx$ $\forall x \in \mathcal{D}(A)$.

Clearly, $\Gamma(A) = \{(x, Ax) : x \in \mathcal{D}(A)\} \subseteq \underbrace{\{(x, Bx) : x \in \mathcal{D}(B)\}}_{= \Gamma(B)}$

But then $\overline{\Gamma(A)} \subseteq \overline{\Gamma(B)} = \Gamma(B)$ since B closed.

So if $(0,y) \in \overline{\Gamma(A)}$, then $(0,y) \in \Gamma(B)$ and thus $y = B0 = 0$. (*)

We can therefore define C with

$$\mathcal{D}(C) := \{ x \in \mathcal{H} : (x,y) \in \overline{\Gamma(A)} \text{ for some } y \in \mathcal{H} \},$$

$$Cx := y,$$

where y is the <u>unique</u> vector such that $(x,y) \in \overline{\Gamma(A)}$.

Indeed, assume that $(x, y_1), (x, y_2) \in \overline{\Gamma(A)}$.

Then there exist sequences $(x_n)_{n\in\mathbb{N}}, (\tilde{x}_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$

s.t. $(x_n, Ax_n) \longrightarrow (x, y_1)$, $(\tilde{x}_n, A\tilde{x}_n) \longrightarrow (x, y_2)$ in $\mathcal{H} \oplus \mathcal{H}$.

Let $z_n := x_n - \tilde{x}_n \in \mathcal{D}(A)$.

$$\text{Then } \underbrace{(z_n, Az_n)}_{\in \Gamma(A)} \overset{A \text{ linear}}{=} (x_n - \tilde{x}_n, Ax_n - A\tilde{x}_n)$$

$$= (x_n, Ax_n) - (\tilde{x}_n, A\tilde{x}_n)$$

$$\longrightarrow (x, y_1) - (x, y_2) = (0, y_1 - y_2)$$

So $(0, y_1 - y_2) \in \overline{\Gamma(A)} \overset{\circledast}{\Longrightarrow} y_1 - y_2 = 0 \Rightarrow y_1 = y_2$.

C is linear: Let $x, \tilde{x} \in \mathcal{D}(C)$, $\lambda \in \mathbb{C}$. Then there exist $y, \tilde{y} \in \mathcal{H}$ s.t. $(x,y), (\tilde{x}, \tilde{y}) \in \overline{\Gamma(A)}$.

$\Rightarrow \exists (x_n)_{n\in\mathbb{N}}, (\tilde{x}_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ s.t. $(x_n, Ax_n) \to (x, y)$
$(\tilde{x}_n, A\tilde{x}_n) \to (\tilde{x}, \tilde{y})$ in $\mathcal{H} \oplus \mathcal{H}$

Define $z_n := x_n + \lambda \tilde{x}_n \in \mathcal{D}(A)$. Then

$$\underbrace{(z_n, Az_n)}_{\in \Gamma(A)} \overset{A \text{ lin.}}{=} (x_n + \lambda\tilde{x}_n, Ax_n + \lambda A\tilde{x}_n) = (x_n, Ax_n) + \lambda(\tilde{x}_n, A\tilde{x}_n)$$

$$\xrightarrow{n\to\infty} (x,y) + \lambda(\tilde{x}, \tilde{y}) = (x + \lambda\tilde{x}, y + \lambda\tilde{y})$$

$\Rightarrow (x + \lambda\tilde{x}, y + \lambda\tilde{y}) \in \overline{\Gamma(A)} \Rightarrow x + \lambda\tilde{x} \in \mathcal{D}(C)$ and

$$C(x+\lambda\tilde{x}) = y+\lambda\tilde{y} = Cx + \lambda C\tilde{x}. \quad \Rightarrow \quad C \text{ linear}.$$

Furthermore, by construction,

$$\Gamma(C) = \{(x, Cx) : x \in \mathcal{D}(C)\} = \overline{\Gamma(A)} \subseteq \Gamma(B)$$

and $\mathcal{D}(A) \subseteq \mathcal{D}(C) \subseteq \mathcal{D}(B)$, $Cx = Ax \quad \forall x \in \mathcal{D}(A)$.

Thus C is a closed extension of A.

Since B was an arbitrary closed extension of A, C is the smallest closed extension.
Comparing the definition of $\overline{\Gamma(A)}$ and $\overline{A}$, one sees that indeed $C = \overline{A}$.

The reverse is easy. If $\overline{\Gamma(A)} = \Gamma(B)$ for some lin op. B on $\mathcal{H}$, then B is closed and an extension of A, so A closable. ∎