Exercise 10

Problem 1

a) $L_{l,m} = x_l p_m - x_m p_l \qquad p_l = -i\partial_l$

$$-\sum_{l<m} L_{l,m}^2 = \sum_{l<m} (x_l\partial_m - x_m\partial_l)^2 = \sum_{l<m}(x_l\partial_m x_l\partial_m - x_l\partial_m x_m\partial_l - x_m\partial_l x_l\partial_m + x_m\partial_l x_m\partial_l) =$$

$$= \sum_{l<m}(x_l^2\partial_m^2 + x_m^2\partial_l^2 - \partial_m x_m x_l\partial_l - \partial_l x_l x_m\partial_m) = \frac{1}{2}\sum_{l\neq m}(x_l^2\partial_m^2 + x_m^2\partial_l^2 - \partial_m x_m x_l\partial_l - \partial_l x_l x_m\partial_m)$$

$$= \frac{1}{2}\sum_{l,m}(x_l^2\partial_m^2 + x_m^2\partial_l^2 - \partial_m x_m x_l\partial_l - \partial_l x_l x_m\partial_m) - \frac{1}{2}\sum_{l=m} x_l^2\partial_m^2 + x_m^2\partial_l^2 - \partial_m x_m x_l\partial_l - \partial_l x_l x_m\partial_m =$$

$$= |x|^2\Delta - (\nabla\cdot x)(x\cdot\nabla) - \sum_l (x_l^2\partial_l^2 - \partial_l x_l^2\partial_l) = |x|^2\Delta - (\nabla\cdot x)(x\cdot\nabla) + \sum_l 2x_l\partial_l =$$

$$= |x|^2\Delta - (\nabla\cdot x)(x\cdot\nabla) + 2x\cdot\nabla$$

$$\Rightarrow \quad -\Delta = \frac{2}{|x|^2}x\cdot\nabla - \frac{(\nabla\cdot x)(x\cdot\nabla)}{|x|^2} + \frac{\sum_{l<m} L_{l,m}^2}{|x|^2}$$

b) $\nabla\cdot x = \sum_l \partial_l x_l = \sum_l (1 + x_l\partial_l) = n + x\cdot\nabla$

$x\cdot\nabla = r\vec{n}_x\cdot\nabla = r\cdot\partial_r$

$$(\nabla\psi,\nabla\psi) = (\psi,-\Delta\psi) = \left(\psi, \frac{2-n}{r}\partial_r\psi\right) - \left(\psi, \frac{r\partial_r(r\partial_r)}{r^2}\psi\right) + \left(\psi, \frac{1}{r^2}\sum_{l<m} L_{l,m}^2\psi\right)$$

$$\int_{\mathbb{R}^+\times\Omega} \bar\psi\, \partial_{rr}(\psi)\, r^{n-1}\,dr\,d\Omega = -\int_{\mathbb{R}^+\times\Omega} |\partial_r\psi|^2 r^{d-1}\,dr\,d\Omega - (n-1)\int_{\mathbb{R}^+\times\Omega} \frac{1}{r}\bar\psi\,\partial_r\psi\, r^{n-1}\,dr\,d\Omega$$

$$(\nabla\psi,\nabla\psi) = (\partial_r\psi, \partial_r\psi) + \left(\psi, \frac{1}{r^2}\sum_{l<m} L_{l,m}^2\psi\right)$$

Using Einstein summation:

$$\partial_{x_j} = \frac{\partial r}{\partial x_j}\frac{\partial}{\partial r} + \frac{\partial\theta_l}{\partial x_j}\frac{\partial}{\partial\theta_l} \qquad r = \sqrt{x_j^2} \qquad \partial_{x_j} r = \frac{x_j}{r} = \frac{2x_j}{2\sqrt{x_j^2}}$$

$$= \frac{x_j}{r}\frac{\partial}{\partial r} + \frac{\partial\theta_l}{\partial x_j}\frac{\partial}{\partial\theta_l}$$

$$|\nabla f|^2 = |\partial_r f|^2 + \left(\frac{\partial\theta_l}{\partial x_j}\partial_{\theta_l}\bar f\, \frac{\partial\theta_m}{\partial x_j}\partial_{\theta_m} f\right) + \left(\frac{\partial\theta_l}{\partial x_j}\partial_{\theta_l}\bar f\, \frac{x_j}{r}\partial_r f\right) + \left(\frac{\partial\theta_l}{\partial x_j}\frac{x_j}{r}\partial_r\bar f\,\partial_{\theta_l} f\right)$$

$$\frac{\partial\theta_l}{\partial x_j}\frac{x_j}{r} = \frac{\partial\theta_l}{\partial x_j}\frac{\partial x_j}{\partial r} = \frac{\partial\theta_l}{\partial r} = 0 \quad , \; x_j = r F(\theta_l)$$

$$|\nabla f|^2 = |\partial_r f|^2 + \left(\frac{1}{r^2}\frac{\partial\theta_l}{\partial F_j(\theta_l)}\partial_{\theta_l}\bar f\,\frac{\partial\theta_m}{\partial F_j(\theta_l)}\partial_{\theta_m} f\right) = |\partial_r f|^2 + \frac{1}{r^2}\frac{\partial\bar f}{\partial F_j(\theta_l)}\frac{\partial f}{\partial F_j(\theta_l)}$$

Problem 2

$$0 \le \|(\beta H_0 - (4\beta)^{-1})\psi\|^2$$

$$0 \le (\psi, (\beta H_0 - (4\beta)^{-1})^2 \psi) = (\psi, (\beta^2 H_0^2 - \frac{H_0}{2} + (4\beta)^{-2})\psi)$$

$$(\psi, H_0 \psi) \le (\psi, (\beta^2 H_0^2 + \frac{H_0}{2} + (4\beta)^{-2})\psi) = (\psi, (\beta H_0 + (4\beta)^{-1})^2 \psi)$$

$$\|\sqrt{H_0}\psi\|^2 \le \|(\beta H_0 + (4\beta)^{-1})\psi\|^2$$

$$\|\sqrt{H_0}\psi\| \le \|(\beta H_0 + (4\beta)^{-1})\psi\| \le \beta \|H_0 \psi\| + (4\beta)^{-1}\|\psi\|$$

Problem 3

$$\chi_R = 0 \quad |x| < R \qquad \chi_R \in C^\infty(\mathbb{R}^n)$$
$$= 1 \quad |x| > 2R \qquad 0 \le \chi_R \le 1 \ \forall x$$

$$E\psi = H\psi$$

$$(\varphi, E\psi) = (\varphi, H\psi)$$

$$\varphi = \zeta^2 \psi \qquad \zeta = e^{F_\varepsilon}\chi_R \qquad F_\varepsilon = \frac{g(|x|)}{1+\varepsilon g(|x|)} \qquad g \ge 0$$

$$E\|\zeta\psi\|^2 = (\psi, \zeta^2 H\psi) = \mathrm{Re}(\zeta^2\psi, H\psi)$$

$$2\mathrm{Re}(\zeta^2\psi, H\psi) = (\psi, \zeta^2 H\psi) + (\zeta^2 H\psi, \psi) = (\psi, (\zeta^2 H + H\zeta^2)\psi)$$

$$H\zeta^2 + \zeta^2 H = [[H, \zeta], \zeta] + 2\zeta H \zeta$$

$$[-\Delta + V, \zeta]f = -\Delta(\zeta f) + \zeta\Delta f = -\nabla\cdot(\nabla(\zeta f)) + \zeta\Delta f = -\nabla(\zeta\nabla f + f\nabla\zeta) + \zeta\Delta f = (-2\nabla f\nabla\zeta - \Delta\zeta) f$$

$$[[-\Delta + V, \zeta], \zeta]f = -2[\nabla\zeta\nabla, \zeta]f = -2(\nabla\zeta^2 f + \zeta\nabla\zeta\nabla f - \zeta\nabla\zeta\nabla f) = -2|\nabla\zeta|^2 f$$

$$\mathrm{Re}(\zeta^2\psi, H\psi) = (\zeta\psi, H\zeta\psi) - (\psi, |\nabla\zeta|^2\psi)$$

$$\nabla\zeta = \nabla(e^{F_\varepsilon}\chi_R) = \nabla e^{F_\varepsilon}\chi_R + e^{F_\varepsilon}\nabla\chi_R = e^{F_\varepsilon}\chi_R\nabla F_\varepsilon + e^{F_\varepsilon}\nabla\chi_R$$

$$\nabla F_\varepsilon = \frac{g'(|x|)}{(1+\varepsilon g)^2}\nabla|x| = \frac{g'}{(1+\varepsilon g)^2}\frac{x}{|x|} \qquad |\nabla\zeta|^2 = \zeta^2|\nabla F_\varepsilon|^2 + e^{2F_\varepsilon}|\nabla\chi_R|^2 + 2\chi_R e^{2F_\varepsilon}\nabla F_\varepsilon\nabla\chi_R$$

$$|\nabla F_\varepsilon|^2 = \frac{g'^2}{(1+\varepsilon g)^2} \le (g'(|x|))^2$$

$$E\|\zeta\psi\|^2 = (\zeta\psi, (\underbrace{-\Delta}_{\le 0} + V - |\nabla F_\varepsilon|^2)\zeta\psi) - (\psi, \underbrace{(e^{2F_\varepsilon}|\nabla\chi_R|^2 + 2\chi_R e^{2F_\varepsilon}\nabla F_\varepsilon\nabla\chi_R)}_{\sup C_R < \infty}\psi)$$

$$E\|\zeta\psi\|^2 \ge (\zeta\psi, (V - |\nabla F_\varepsilon|^2)\zeta\psi) - \sup C_R$$

$$E\|\zeta\psi\|^2 \geq (\zeta\psi, [V - g'(|x|)^2]\zeta\psi) - \sup C_R$$

$$\sup C_R \geq (\zeta\psi, (V - g'^2 - E)\zeta\psi)$$

$$0 < \inf_{R > R_0} (V - g'^2 - E)$$

precise choice $g'(|x|)^2 = V - E - \delta \Rightarrow \|\zeta\psi\|^2 \leq \frac{\sup C_R}{\delta}$

$$g'(r) = \sqrt{Cr^2 - d}$$

$$g = \frac{1}{2} r\sqrt{Cr^2 - d} + C_1 - \frac{d \ln(Cr + \sqrt{C}\sqrt{Cr^2 - d})}{2\sqrt{C}}$$

alternatively $g'(|x|) = \sqrt{C - \delta}\, r$

$$g(r) = \frac{\sqrt{C-\delta}}{2} r^2$$

to conclude $x \to e^{g} \cdot \psi(x) \in L^2$

<u>Problem 4</u>

same approach as in 3 up to $0 < \inf_{R > R_0} (V - g'^2 - E)$

$$g'(r) = \sqrt{Cr^\alpha - d}$$

← Hypergeometric function

$$g(r) = C_1 - \frac{r\left(2d - 2Cr^\alpha + \sqrt{d}\sqrt{d - Cr^\alpha}\,\alpha\, {}_2F_1\left(\frac{1}{2}, \frac{1}{\alpha}, 1 + \frac{1}{\alpha}, \frac{Cx^\alpha}{d}\right)\right)}{\sqrt{Cr^\alpha - d}\; 2 + \alpha}$$

alternatively

$$g'(r) = \sqrt{C - \delta}\, r^{\frac{\alpha}{2}}$$

$$g(r) = \frac{2}{2+\alpha}\sqrt{C-\delta}\, r^{\frac{\alpha}{2}+1}$$