Exercise 11

Problem 1

$R = R_A(z) \Rightarrow$

$$\left((A-z)^*\phi, R_A(z)\psi\right) = \left((R_A(z))^*(A-z)^*\phi, \psi\right) = \left([(A-z)R_A(z)]^*\phi, \psi\right) = (\phi, \psi)$$

$\Leftarrow$: A closed, $A^{**} = A \Rightarrow (A-z)^{**} = A - z$

$$\mathcal{D}\left(((A-z)^*)^*\right) = \left\{\tilde{\psi} \in \mathcal{H} \mid \exists \psi \in \mathcal{H} : \left((A-z)^*\varphi, \tilde{\psi}\right) = (\varphi, \psi), \forall \varphi \in \mathcal{D}(A^*)\right\}$$

$$\left\{\tilde{\psi} \in \mathcal{H} \mid \mathcal{D}(A^*) \ni \varphi \to \left((A-z)^*\varphi, \tilde{\psi}\right) \text{ extends to a continuous bounded map on } \mathcal{H}\right\}$$

$$\left((A-z)^*\varphi, \underbrace{R\psi}_{\tilde{\psi}}\right) = (\varphi, \psi)$$

$$\Rightarrow R\psi \in \mathcal{D}\left(((A-z)^*)^*\right) = \mathcal{D}(A^{**}) = \mathcal{D}(A) \Rightarrow \left((A-z)^*\varphi, R\psi\right) = \left(\varphi, (A-z)R\psi\right) = (\varphi, \psi)$$

$(A-z)R = \mathbb{1}$

$R = (A-z)^{-1}$

Problem 2

$$|\psi(x)|^2 = \int_{-\infty}^{x} \frac{d}{dy}\left(|\psi(y)|^2\right) dy \leq \int_{-\infty}^{\infty} 2|\psi'(y)||\psi(y)| dy \leq \varepsilon^2 \|\psi'\|_2^2 + \frac{1}{\varepsilon^2}\|\psi\|_2^2$$

Problem 3

$$\int_{\mathbb{R}} |V(x)||\psi(x)|^2 dx \leq \|V_1\|_1 \|\psi\|_\infty + \|V_\infty\|_\infty \|\psi\|_2 \leq 2\varepsilon^2\|\psi'\|^2 + \left(\frac{2}{\varepsilon^2} + \|V_\infty\|_\infty\right)\|\psi\|^2$$

$$V(x) = \underset{L^\infty(\mathbb{R})}{\underset{\cap}{V_\infty(x)}} + \underset{L^1(\mathbb{R})}{\underset{\cap}{V_1(x)}}$$

Problem 4

$$|\psi(0_+) - \psi_0(0_-)|^2 \leq |\psi(0_+)|^2 + |\psi(0_-)|^2 + 2|\psi(0_+)||\psi(0_-)| \leq 2|\psi(0_+)|^2 + |\psi(0_-)|^2$$

$$|\psi(0_+)|^2 = \int_\infty^0 \frac{d}{dx}|\psi(x)|^2 dx \leq \varepsilon\|\psi'\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\varepsilon}\|\psi\|^2_{L^2(\mathbb{R}^+)}$$

$$|\psi(0_-)|^2 \leq \varepsilon\|\psi'\|^2_{L^2(\mathbb{R}^-)} + \frac{1}{\varepsilon}\|\psi\|^2_{L^2(\mathbb{R}^-)}$$

$$|\psi(0_+) - \psi(0_-)|^2 \leq \varepsilon\|\psi'\|^2 + \frac{1}{\varepsilon}\|\psi\|^2$$

Problem 5

a) $V = \underset{\in L^p}{V_p} + \underset{\in L^\infty}{V_\infty}$

$$(\psi, V\psi) = (\psi, V_\infty \psi) + (\psi, V_p \psi) = \|V_\infty\|_\infty \|\psi\|_2^2 + (\psi, V_p \psi)$$

$$\int |V(x)| |\psi(x)|^2 \, dx \le \|V\|_p \, \| |\psi|^2 \|_r \qquad \frac{1}{p} + \frac{1}{r} = 1$$
$$\le \|V\|_p \, \|\psi\|_{2r}^2$$

$$\|\psi\|_q = \|\mathcal{F}^{-1} \hat{\psi}\|_q \le \|\hat{\psi}\|_p \qquad \frac{1}{q} + \frac{1}{p} = 1 \quad (\text{H.-Y. inequality}) \quad q \ge 2$$

$$\|\hat{\psi}\|_{2s} = \left(\int |\hat{\psi}|^{2s} d\eta\right)^{\frac{1}{2s}} = \left(\int (1+|\eta|^2)^{-s} (1+|\eta|^2)^{s} |\hat{\psi}|^{2s} d\eta\right)^{\frac{1}{2s}} \le$$
$$\le \|(1+|\eta|^2)^{-s}\|_m^{\frac{1}{2s}} \, \|(1+|\eta|^2)^{s} |\hat{\psi}|^{2s}\|_n^{\frac{1}{2s}} \qquad \frac{1}{m} + \frac{1}{n} = 1$$

$$\|(1+|\eta|^2)^{s} |\hat{\psi}|^{2s}\|_{\frac{1}{s}} = \|(1+|\eta|^2) |\hat{\psi}|^2\|_1^{s} = \left(\int (1+|\eta|^2) |\hat{\psi}|^2 d\eta\right)^{s} = \left(\|\hat{\psi}\|_2^2 + \|\eta \hat{\psi}\|_2^2\right)^{s}$$

$$\frac{1}{m} = 1 - \frac{1}{n} = 1 - s$$

$$\|(1+|\eta|^2)^{-s}\|_m = \|(1+|\eta|^2)^{-s}\|_{\frac{1}{1-s}} = \left(\int \left(\frac{1}{(1+|\eta|^2)^{s}}\right)^{\frac{1}{1-s}} d\eta\right)^{1-s} = C$$

$$\int \left(\frac{1}{1+|\eta|^2}\right)^{\frac{s}{1-s}} < \infty \iff \eta^{-\frac{2s}{1-s} + d - 1} < \eta^{-1} \Rightarrow d < \frac{2s}{1-s} \iff s > \frac{d}{2+d}$$

$$\|\psi\|_q \le \|\hat{\psi}\|_{2s} \le C^{\frac{1}{2s}} \left(\|\hat{\psi}\|_2^2 + \|\eta \hat{\psi}\|_2^2\right)^{\frac{1}{2}} \le C^{\frac{1}{2s}} \left(\|\hat{\psi}\|_2 + \|\eta \hat{\psi}\|_2\right) \qquad \sqrt{a^2 + b^2} \le a + b$$
$$\frac{1}{q} = 1 - \frac{1}{2s} > 1 - \frac{d+2}{2d} = \frac{d-2}{2d}$$
$$2 \le q < \frac{2d}{d-2}$$

b) $\hat{\psi}_\delta(\eta) = \delta^{\frac{d}{p}} \hat{\psi}(\delta \eta)$

$$\|\hat{\psi}_\delta\|_p = \|\hat{\psi}\|_p$$
$$\| |\eta| \hat{\psi}_\delta \|_2 = \left(\int |\eta|^2 |\hat{\psi}(\delta\eta)|^2 \delta^{\frac{2d}{p}} d\eta\right)^{\frac{1}{2}} = \delta^{\frac{d}{p} - 1 - \frac{d}{2}} \| |\eta| \hat{\psi} \|_2$$
$$\|\hat{\psi}_\delta\|_2 = \delta^{\frac{d}{p} - \frac{d}{2}}$$

$$\|\psi\|_q \le C^{\frac{1}{2s}} \delta^{\frac{d}{p} - \frac{d}{2}} \left(\delta^{-1} \| |\eta| \hat{\psi} \|_2 + \|\hat{\psi}\|_2\right)$$
$$\frac{d}{p} - \frac{d}{2} = \frac{d(2-p)}{2p} = \frac{d(1-s)}{2s} < 1 \Rightarrow 0 < C^{\frac{1}{2s}} \delta^{\frac{d}{p} - \frac{d}{2} - 1} \ll 1 \text{ for } \delta \gg 1$$

c) $\|V\|_r \qquad \frac{1}{r} = 1 - \frac{1}{s} < 1 - \frac{d-2}{d} = \frac{2+d-d}{d} \Rightarrow r > \frac{d}{2}$