Exercise 12

Problem 1

$$(\varphi, C_q\varphi) = q\left((A+\lambda)^{-\frac{1}{2}}\varphi\right) \leq a\left((A+\lambda)^{-\frac{1}{2}}\varphi, (A-\gamma)(A+\lambda)^{-\frac{1}{2}}\varphi\right) + b\|(A+\lambda)^{-\frac{1}{2}}\varphi\|^2$$

$$\leq a\left((A+\lambda)^{-\frac{1}{2}}\varphi, (A+\lambda)(A+\lambda)^{-\frac{1}{2}}\varphi\right) - a(\varphi, \lambda(A+\lambda)^{-1}\varphi) - a(\varphi, \gamma(A+\lambda)^{-1}\varphi) + b(\varphi, (A+\lambda)^{-1}\varphi)$$

$$= a(\varphi,\varphi) + (\varphi, [b - a(\lambda+\gamma)](A+\lambda)^{-1}\varphi) = *$$

$\lambda+\gamma > 0$

$b > a(\lambda+\gamma) \Rightarrow \frac{b-a(\lambda+\gamma)}{A+\lambda} > 0 \Rightarrow \frac{b-a(\lambda+\gamma)}{A+\lambda} \leq \frac{b}{\gamma+\lambda} - a$

$* \leq a(\varphi,\varphi) + (\varphi, \frac{b}{\gamma+\lambda} - a\varphi) = \frac{b}{\gamma+\lambda}\|\varphi\|^2$

$b < a(\lambda+\gamma) \Rightarrow \frac{b-a(\lambda+\gamma)}{A+\lambda} < 0 \Rightarrow \frac{b-a(\lambda+\gamma)}{A+\lambda} \leq 0$

$* \leq a(\varphi,\varphi)$

Problem 2

a) $|q(\varphi)| \leq |q((A+\lambda)^{-\frac{1}{2}}(A+\lambda)^{\frac{1}{2}}\varphi)| = |q((A+\lambda)^{-\frac{1}{2}}\psi)| = |(\psi, C_q(\lambda)\psi)| \leq a\|\psi\|^2 \leq$

$\leq a(\varphi, (A+\lambda)\varphi) = a(\varphi, (A-\gamma)\varphi) + a(\lambda+\gamma)\|\varphi\|^2$

b) from 2a we take $\lambda_0 \gg 1$ and $a_{\lambda_0} = \|C_q(\lambda_0)\|$

furthermore $|q(\varphi)| \leq a_{\lambda_0} q_{A-\gamma}(\varphi) + b_{\lambda_0}\|\varphi\|^2$

using 1 $\quad \|C_q(\lambda)\| \leq \max\left(a_{\lambda_0}, \frac{b_{\lambda_0}}{\lambda+\gamma}\right) \quad \forall \lambda > \gamma$

$\limsup_{\lambda\to\infty} \|C_q(\lambda)\| \leq a_{\lambda_0} = \|C_q(\lambda_0)\| \Rightarrow \limsup_{\lambda\to\infty} \|C_q(\lambda)\| \leq \liminf_{\lambda_0\to\infty} \|C_q(\lambda_0)\|$

$\Rightarrow \lim_{\lambda\to\infty} \|C_q(\lambda)\|$ exists and it is finite

$\forall \varepsilon > 0 \quad a = a_0 + \varepsilon \quad \exists\, b$ such that (1) holds

$\|C_q(\lambda)\| \leq \max\left(a, \frac{b}{\lambda+\gamma}\right)$

$\lim_{\lambda\to\infty} \|C_q(\lambda)\| \leq a = a_0 + \varepsilon \qquad \|C_q(\lambda)\| \geq a_0$

$a_0 \leq \lim_{\lambda\to\infty} \|C_q(\lambda)\| \leq a_0$