Exercise 13

Problem 1

spherical coordinates such that

$x_1 = r\cos\varphi\cos\theta \qquad R_1 = 0$

$x_2 = r\sin\varphi\cos\theta \qquad R_2 = 0$

$x_3 = r\sin\theta \qquad R_3 = R$

$$\int_{|x|=r}\frac{d\sigma(x)}{|x-R|} = \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r^2\cos\theta\, d\theta\, d\varphi}{(R^2+r^2-2rR\sin\theta)^{\frac{1}{2}}} = 2\pi r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta\, d\theta}{(R^2+r^2-2rR\sin\theta)^{\frac{1}{2}}} = \begin{bmatrix} y = \sin\theta \\ dy = \cos\theta\, d\theta\end{bmatrix} =$$

$$= 2\pi r^2\int_{-1}^{1}\frac{dy}{\sqrt{R^2+r^2-2rRy}} = \left[-2\pi r^2\frac{2}{2rR}\sqrt{R^2+r^2-2rRy}\right]_{-1}^{1} = 4\pi r^2\,\frac{R+r-|R-r|}{2rR} = \frac{4\pi r^2}{\max\{r,R\}}$$

Problem 2

$E = a + k^2 \qquad k\in\mathbb{R}^3$

test function

$\psi_n(x,y) = e^{i\vec{k}\vec{x}}\varphi_n(\vec{x},\vec{y}) \qquad \varphi_n = \left(\frac{1}{\sqrt{n}}\right)^3\varphi\left(\frac{\vec{x}}{n}-\vec{y}\right) \qquad \|\varphi\| = 1 \quad \varphi\in C_0^\infty(\mathbb{R}^3)$

$\|\psi_n\| = 1 \qquad \varphi(x) = 1 \quad x\in B_1(0) \qquad \mathrm{supp}(\varphi) = B_2(0)$

$$\|(-\Delta+V-E)\psi_n\| \le \|(-\Delta-k^2)\psi_n\| + \|(V-a)\psi_n\|$$

$$-\Delta\psi_n = \varphi_n(-\Delta\exp(i\vec{k}\vec{x})) + \exp(i\vec{k}\vec{x})(-\Delta\varphi_n) - 2\nabla\exp(i\vec{k}\vec{x})\nabla\varphi_n$$

$$= k^2\psi_n(x,y) + \exp(i\vec{k}\vec{x})\frac{1}{n^2}(-\Delta\varphi)\left(\frac{\vec{x}}{n}-\vec{y}\right)\left(\frac{1}{\sqrt{n}}\right)^3 - 2\frac{1}{n}\exp(i\vec{k}\vec{x})\vec{k}(\nabla\varphi)\left(\frac{\vec{x}}{n}-\vec{y}\right)\cdot\left(\frac{1}{\sqrt{n}}\right)^3$$

$$\|(-\Delta-k^2)\psi_n\| \le \frac{1}{n^2}\|-\Delta\varphi\|_\infty\underbrace{\left\|\left(\frac{1}{\sqrt{n}}\right)^3\chi_{\mathrm{supp}\,\varphi_n}\right\|_2}_{\le\frac{4}{3}\pi} + \frac{2|k|}{n}\|\nabla\varphi\|_\infty\underbrace{\left\|\left(\frac{1}{\sqrt{n}}\right)^3\chi_{\mathrm{supp}\,\varphi_n}\right\|_2}_{\le\frac{4}{3}\pi} \xrightarrow[n\to\infty]{} 0$$

$\lim_{|x|\to\infty} V = a$

$\Downarrow$

$\lim_{|x|\to\infty}(V(x)-a) = 0 \Rightarrow \exists r_n\ \forall|x|>r_n : (V(x)-a) < \frac{1}{n}$

$\|(V-a)\psi_n\| < \frac{1}{n} \qquad$ for $\vec{y} = \vec{e}\cdot(2n+r_n) \quad |\vec{e}| = 1$ i.e. $\mathrm{supp}(\varphi_n(x,y)) \subset B_{r_n}(0)^c$

$\Downarrow$

$\|(-\Delta+V-E)\psi_n\| \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \sigma_{ess} \supset [a,\infty)$

Now we need to show $\lambda \in \sigma_{ess}(H) \Rightarrow \lambda \geq a$

$\lambda \in \sigma_{ess}(H)$

Weyl sequence: $\|\psi_n\| = 1$, $\psi_n \in \mathcal{D}(H)$, $\psi \xrightarrow{w} 0$, $\|(H-\lambda)\psi_n\| \to 0$

$\lambda = \lim_{n\to\infty} (\psi_n, H\psi_n)$

$f_0(x) \geq 0$ $\quad f_0(x) = 1 \quad |x| \leq 1$ $\quad\quad f_1(x)$ $\quad f_1(x) = 0 \quad |x| \leq 1$

$f_0(x) = 0 \quad |x| \geq 2$ $\quad\quad f_1(x) = 1 \quad |x| \geq 2$

$f_0(x) > 0 \quad |x| < \frac{7}{4}$ $\quad\quad f_1(x) > 0 \quad |x| > \frac{5}{4}$

$f_1(x)^2 + f_0(x)^2 \geq C > 0 \quad \forall x$

$J_l(x) = \frac{f_l(x)}{(f_0(x)^2 + f_1(x)^2)^{\frac{1}{2}}} \quad l = 0,1$

$J_{l,R}(x) = J_l\left(\frac{x}{R}\right)$

using IMS $\Rightarrow$ $H = J_{1,R} H J_{1,R} + J_{0,R} H J_{0,R} - \sum_{l=0}^{1} |\nabla J_{l,R}|^2$ as quadratic forms

$$\Rightarrow (\psi_n, H\psi_n) = \underbrace{(\psi_n, J_{1,R} H J_{1,R}\psi_n)}_{\geq \Sigma_R \|J_{1,R}\psi_n\|^2} + \underbrace{(\psi_n, J_{0,R} H J_{0,R}\psi_n)}_{\geq E_0 (\psi_n, J_{0,R}^2 \psi_n)} - \sum_{l=0}^{1} (\psi_n, \underbrace{|\nabla J_{l,R}|^2}_{\leq \frac{C}{R^2}} \psi_n)$$

$$(\psi_n, H\psi_n) \geq \Sigma_R + (E_0 - \Sigma_R)(\psi_n, J_{0,R}^2 \psi_n) - \sum_{l=0}^{1} (\psi_n, |\nabla J_{l,R}|^2 \psi_n)$$

$$J_{0,R}\psi_n = \underbrace{J_{0,R}(H+i)^{-1}}_{\text{compact}} \underbrace{(H+i)\psi_n}_{\xrightarrow{w} 0} \to 0 \qquad (H+i)\psi_n = \underbrace{(H-\lambda)\psi_n}_{\to 0} + \underbrace{(i+\lambda)\psi_n}_{\xrightarrow{w} 0} \xrightarrow{w} 0$$

$\lambda \geq \Sigma_R - \frac{C}{R^2} \quad \forall R > 0 \quad \Rightarrow \lambda \geq \Sigma$

$\varphi_R \in \mathcal{D}(H)$ $\operatorname{supp} \varphi_R \subset B_R(0)^c$

$H\varphi_R = -\Delta\varphi_R + V\varphi_R \geq V\varphi_R \quad \Rightarrow \Sigma_R(H) \geq \Sigma_R(V)$

choose R_ε such that $|V(x) - a| < \frac{1}{\varepsilon}$ a.e. $x \in B_{R_\varepsilon}(0)^c$

$$\int_{B_{R_\varepsilon}(0)^c} V(x) |\varphi_{R_\varepsilon}|^2 dx \geq \inf_{B_{R_\varepsilon}(0)^c} V(x) \|\varphi_{R_\varepsilon}\|^2 \geq \left(a - \frac{1}{\varepsilon}\right) \|\varphi_{R_\varepsilon}\|^2 \Rightarrow \Sigma_{R_\varepsilon}(H) \geq a - \frac{1}{\varepsilon}$$

$\Rightarrow \Sigma(H) \geq a$

$\Rightarrow \lambda \geq a \Rightarrow \sigma_{ess}(H) = [a, \infty)$