Exercise 1)

a) proof by contradiction, assume $\dim \mathcal{H} = n < \infty$

$\dim \mathcal{H} < \infty \Rightarrow P, Q$ bounded ; $\mathcal{D}(P) = \mathcal{D}(Q) = \mathcal{H}$; P, Q are matrices

$PQ - QP = \frac{1}{i} I$

$\operatorname{tr} PQ - \operatorname{tr} QP = \frac{1}{i} \operatorname{tr} I = \frac{n}{i}$ (1)

$\operatorname{tr} PQ = \sum_{j=1}^{n} \langle \varphi_j, PQ\varphi_j \rangle = \sum_{j,k} \langle \varphi_j, P\varphi_k \rangle \langle \varphi_k, Q\varphi_j \rangle = \operatorname{tr} QP$

(1) $\Rightarrow 0 = \frac{n}{i}$ contradiction

b) proof by contradiction, assume $\exists f \in \mathcal{H}$ $Pf = \lambda f$ $\lambda \in \mathbb{R}$, $\|f\| \neq 0$

$\langle Pf, Qf \rangle - \langle Qf, Pf \rangle = \frac{1}{i} \|f\|^2$

$\lambda \langle f, Qf \rangle - \lambda \langle Qf, f \rangle = \frac{1}{i} \|f\|^2$

$0 = \frac{1}{i} \|f\|^2$ contradiction

for Q analogically

c) proof by contradiction, assume P bounded [BEH08] [Proposition 5.4.1]

$\lambda \in \sigma(Q) \subset \mathbb{R}$ from: Thm: Spectrum of s.a. operator is non-empty

$\exists (\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(Q), \|\varphi_n\| = 1, \|(Q-\lambda)\varphi_n\| \to 0$ from: Thm Weyl criterion

[BEH08] [Corollary 4.3.5]

$\frac{1}{i} \|\varphi_n\|^2 = \langle P\varphi_n, Q\varphi_n \rangle - \langle Q\varphi_n, P\varphi_n \rangle$

$1 = | \langle P\varphi_n, Q\varphi_n \rangle - \langle Q\varphi_n, P\varphi_n \rangle |$

$1 = | \langle P\varphi_n, (Q-\lambda)\varphi_n \rangle - \langle (Q-\lambda)\varphi_n, P\varphi_n \rangle |$

$1 \leq 2 \|P\varphi_n\| \, \|(Q-\lambda)\varphi_n\|$

$1 \leq 2 \|P\| \, \|(Q-\lambda)\varphi_n\| \xrightarrow[n\to\infty]{} 0$

Exercise 2)

b) Def.: Regularity domain $\pi(T)$: $T \in \mathcal{C}(X)$. $\pi(T) = \{ \lambda \in \mathbb{C} \mid \forall x \in D_T \; \exists c(\lambda) > 0 : \|(T-\lambda)x\| \geq c_\lambda \|x\| \}$

we show that $\mathbb{C}^{\pm}$ is in regularity domain of A

$\lambda \in (\mathbb{C} \setminus \mathbb{R}) \Leftrightarrow \lambda = \nu + i\mu \;;\; \nu, \mu \in \mathbb{R}, \mu \neq 0$

[BEH08]: J. Blank, M. Havlíček, P. Exner: Hilbert space operators in quantum physics, 2008

$\|(A-\lambda)\varphi\| \geq |\mu|^2 \|\varphi\|^2$ due to the fact that $\sigma(A) \subset \mathbb{R}$

$\Pi(A) \supset \mathbb{C}^+ \cup \mathbb{C}^-$

proof is completed by Thm.: map $\lambda \mapsto \operatorname{def}(T-\lambda) = \dim \operatorname{Ker}(T^* - \bar{\lambda})$ is constant on any connected component of regularity domain [BEH08][Theorem 4.7.1]

a) we show that a) is implied by b) for ~~fixed~~ $d_\pm = 0$

$\operatorname{Ran}(A-\lambda) = \mathcal{H} \Leftrightarrow \{0\} = \operatorname{Ran}(A-\lambda)^\perp = \operatorname{Ker}(A^* - \bar{\lambda}) \Rightarrow$

$\dim(\operatorname{Ker}(A^* - \mu_0)) = 0$ and using symmetry of A and previous Theorem gives us desired result

Exercise 3)

a) $\mathcal{D}(AB) = \{\psi \in \mathcal{H} \mid B\psi \in \mathcal{D}(A)\}$

$\tau_n \to \tau \qquad \tau_n \in \mathcal{D}(AB)$

$\|\tau_n - \tau\| < \varepsilon \Rightarrow \|B\tau_n - B\tau\| < \varepsilon\|B\| \Rightarrow B\tau_n$ Cauchy + $\mathcal{D}(A)$ closed $\Rightarrow B\tau \in \mathcal{D}(AB)$

from def. of A is closed $A(B\tau_n) \to \varphi \Rightarrow A(B\tau) = \varphi$ we rewrite it as $(AB)\tau_n \to \varphi \Rightarrow (AB)\tau = \varphi$

b) $\mathcal{D}(BA) = \mathcal{D}(A) \Rightarrow \underset{\mathcal{D}(BA)}{\varphi_n} \to \varphi \Rightarrow \varphi \in \mathcal{D}(BA)$

we have $A\varphi_n \to \psi \Rightarrow BA\varphi_n \to B\psi = \tau$

$\|BA\varphi_n - \tau\| < \varepsilon \Rightarrow \|A\varphi_n - B^{-1}\tau\| < \varepsilon\|B^{-1}\| \Rightarrow A\varphi = \psi$ and $BA\varphi = \tau$

Exercise 4) closedness:

$\psi_n \to \psi$ in L^2 implies that we can choose convergent subsequence which converge pointwise $d\mu$-almost everywhere

$\psi_n(x) \to \psi(x)$ $d\mu$ a.e.

$V(x)\psi_n(x) \to V(x)\psi(x)$ $d\mu$ a.e.
$(V\psi_n)(x) \to \phi(x)$ $d\mu$ a.e. $\Rightarrow V\psi = \phi$ $d\mu$ a.e.

$\|M\psi\| \leq \underbrace{\|V\psi_n\|}_{<\infty} + \underbrace{\|V\psi_n - V\psi\|}_{<\varepsilon} \Rightarrow \psi \in \mathcal{D}(M_V)$

spectrum:

inverse of M_V: $[(M_V - \lambda)^{-1}\psi](x) = \left(\frac{1}{V-\lambda}\psi\right)(x)$

$\lambda \notin \operatorname{essran} V \Leftrightarrow \exists \varepsilon > 0 : d\mu$ a.e. $x \in \mathbb{R}^d$ $|V(x) - \lambda| \geq \varepsilon$

$\Rightarrow \left(\lambda \notin \operatorname{essran} V \Rightarrow \frac{1}{M_V - \lambda} \text{ is bounded} \Rightarrow \lambda \in \rho(M_V)\right)$

$\Rightarrow \sigma(M_V) \subset \operatorname{essran} V$

$(M_V - \lambda)^{-1}$ is bounded $\Leftrightarrow \operatorname{esssup}_{x \in \mathbb{R}^d} \left|\frac{1}{V(x)-\lambda}\right| < \infty$

opposite inclusion:

$\lambda \in \operatorname{essup} V \Rightarrow \forall \varepsilon > 0$ $\mu(\{x \mid |V(x) - \lambda| < \varepsilon\}) > 0$

$\mu\left(\{x \mid \frac{1}{\varepsilon} < \frac{1}{|V(x)-\lambda|}\}\right) > 0$

$\Rightarrow \operatorname{esssup} \frac{1}{|V(x)-\lambda|} = +\infty$ $(M_V - \lambda)^{-1}$ is not bounded

Exercise 2) alternative full proof of b)

$\lambda = \nu + i\mu$, $\mu \neq 0$, $\nu, \mu \in \mathbb{R}$

A symmetric $\Rightarrow \|(A-\lambda)\varphi\|^2 \geq \mu^2 \|\varphi\|^2 \quad \forall \varphi \in \mathcal{D}(A) \subset \mathcal{H}$ (1)

A closed + (1) $\Rightarrow$ Ran $(\lambda - A)$ is closed subspace of $\mathcal{H}$

$\mathrm{Ker}(\lambda - A^*) = \mathrm{Ran}(\bar{\lambda} - A)^\perp$ (2)

we want to show that $\dim \mathrm{Ker}(\lambda - A^*)$ is locally constant in $\mathbb{C}^\pm$

η small $\eta \in \mathbb{C}$ $\quad \dim \mathrm{Ker}((\lambda+\eta) - A^*) \overset{?}{=} \dim \mathrm{Ker}(\lambda - A^*)$

$u \in \mathcal{D}(A^*)$, $\|u\| = 1$, $u \in \mathrm{Ker}((\lambda+\eta) - A^*)$

we suppose that $(u, v) = 0$ $\quad v \in \mathrm{Ker}(\lambda - A^*)$ and show contradiction

using (2) $\Rightarrow u \in \mathrm{Ran}(\bar{\lambda} - A) \Rightarrow \varphi \in \mathcal{D}(A)$ $\quad (\bar{\lambda} - A)\varphi = u$

$$0 = \underbrace{(((\lambda+\eta) - A^*)u, \varphi)}_{\uparrow\, u \in \mathrm{Ker}((\lambda+\eta) - A^*)} = (u, (\bar{\lambda} - A)\varphi) + \bar{\eta}(u, \varphi) = \|u\|^2 + \bar{\eta}(u, \varphi)$$

This is a contradiction for $|\eta| < |\mu|$ because (1) implies $\|u\| \geq |\mu| \|\varphi\|$

We can conclude that for $|\eta| < |\mu|$ there is no $u \in \mathrm{Ker}((\lambda+\eta) - A^*)$ which is in $\mathrm{Ker}(\lambda - A^*)^\perp$. This implies that $\dim \mathrm{Ker}((\lambda+\eta) - A^*) \leq \dim \mathrm{Ker}(\lambda - A^*)$

We can use the same argumentation for $|\eta| < \frac{|\mu|}{2}$ to show that

$\dim \mathrm{Ker}(\lambda - A^*) \leq \dim \mathrm{Ker}((\lambda+\eta) - A^*)$

Put together we have $\quad \dim \mathrm{Ker}(\lambda - A^*) = \dim \mathrm{Ker}((\lambda+\eta) - A^*)$ for $|\eta| < \frac{|\mu|}{2}$

By this we proved that $\dim \mathrm{Ker}(\lambda - A^*)$ is locally constant. Because we had no special conditions on λ we can conclude that the dimension is constant on upper half-plane of $\mathbb{C}$ and on lower half-plane of $\mathbb{C}$. We note that these dimensions might not be equal.