Exercise 1)

a) Proof by contradiction, assume $\dim X = m < \infty$

$\dim X < \infty \Rightarrow P/Q$ bounded, $D(\lambda) = D(\alpha) = X$, $P, Q$ are matrices

$PQ - QP = \frac{4}{\pi} J$

$\lambda Q P - \lambda N Q P = \frac{4}{\pi} \lambda N Q P = \frac{m}{\pi} \lambda N Q P$ \hspace{1cm} (1)

$\lambda N P Q = \sum_{i=1}^{m} \langle \varphi_i, P Q \varphi_i \rangle = \sum_{i=1}^{m} \langle \varphi_i, P \varphi_i \rangle \langle \varphi_i, Q \varphi_i \rangle = \lambda N Q P$

(1) $\Rightarrow 0 = \frac{m}{\pi} \lambda N Q P$

b) Proof by contradiction, assume $\exists f \in X$, $P f = \lambda f$, $\lambda \in \mathbb{C}$, $|\lambda| \neq 0$

$\langle P g, f \rangle - \langle Q g, f \rangle = \frac{4}{\pi} \| f \|^2$

$\langle f, Q f \rangle - \langle Q f, f \rangle = \frac{4}{\pi} \| f \|^2$

$0 = \frac{4}{\pi} \| f \|^2$ contradiction

for $Q$ analogically

c) Proof by contradiction, assume $P$ bounded

$\lambda \in \mathbb{C}$ from: Thm: Spectrum of a.a. operator is non-empty

$\exists (f_n)_{n \in \mathbb{N}} \in X$ s.t. $\| f_n \|^2 = \| f_n \|^2 - \| Q f_n \|^2$ from: The Weyl criterion

$\frac{1}{\pi} \| f_n \|^2 = \langle P f_n, Q f_n \rangle - \langle Q f_n, P f_n \rangle$

$\frac{1}{\pi} = \| P f_n \|^2 - \| Q f_n \|^2$

$\frac{1}{\pi} = \| P f_n \|^2 - \| Q f_n \|^2$

$\frac{1}{\pi} \leq 2 \| P f_n \|^2 \| (Q - \lambda) f_n \|^2$

$\frac{1}{\pi} \leq 2 \| P f_n \|^2 \| Q f_n \|^2 \rightarrow 0$

Exercise 2)

b) Def: Regularity domain $\pi(\pi) : T \in C(\mathbb{X})$, $\pi(\pi) = \{ \lambda \in \mathbb{C} \mid \forall x \in D, \exists x, x \in \mathbb{X}, \| (T - \lambda) x \| < 1 \}$

we show that $\mathbb{C}^+$ is a regularity domain of $A$

$\lambda \in (\mathbb{C} \setminus \mathbb{R}) \Leftrightarrow \lambda = \nu + i \mu$ where $\nu, \mu \in \mathbb{R}$, $\mu \neq 0$
\[ \|A - \lambda I\| \geq 1/n^2 \|A\|^2 \quad \text{due to the fact that } \sigma(A) = \mathbb{R} \]

\[ T(A) \supset C^+ \cup C^- \]

Proof is completed by showing that \( \lambda \mapsto \text{def}(T - \lambda) = \dim \ker (T^* - \lambda) \) is constant on any connected component of the regularity domain. [BEH08] [Theorem 4.7.1]

a) We show that \( a_0 \) is implied by \( b_0 \) for \( \delta = 0 \nabla (A - \lambda) = 2 \lambda \mapsto \text{def}(A - \lambda) = \ker (A^* - \lambda) \nabla \dim (\ker (A^* - \lambda)) = 0 \) and many symmetric of \( A \) and previous theorem gives the desired result.

Exercise 3)

a) \( \delta(AB) = \{ \psi \in \mathcal{H} \mid B \psi \in \delta(A) \} \]

\[ \gamma_m \rightarrow \gamma \]

\[ \gamma_m \in \delta(AB) \]

\[ \|\gamma_m - \gamma\| < \varepsilon \Rightarrow \|B \gamma_m - B \gamma\| < \varepsilon \Rightarrow B \gamma_m \text{ (closely)} + \delta(A) \Rightarrow B \gamma \in \delta(AB) \]

from def. of \( A \) is closed \( A(B \gamma_m) \rightarrow \psi \Rightarrow AB(\gamma) = \psi \) maximum \( \beta \) as \( (AB) \gamma_m \gamma \Rightarrow (AB) \gamma = \psi \)

b) \( \delta(BA) = \delta(A) \Rightarrow \gamma_m \rightarrow \gamma \Rightarrow \gamma \in \delta(BA) \)

induction \( \delta \gamma_m \rightarrow \psi \Rightarrow BA \gamma_m \rightarrow B \psi = \gamma \)

\[ \|B \gamma_m - \gamma\| < \varepsilon \Rightarrow \|A \gamma_m - B^{-1} \gamma\| < \varepsilon \|B^{-1}\| \Rightarrow A \gamma = \psi \text{ and } BA \gamma = \psi \]
Exercise 4) compactness:
\[ \psi_n \to \psi \text{ in } L^2 \text{ implies that we can choose a subsequence which converges pointwise dy-almost everywhere} \]
\[ \psi_n(x) \to \psi(x) \text{ dy-a.e.} \]
\[ \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \| \psi_n(x) - \psi(x) \| < \epsilon \text{ for all } n \geq N \]
\[ \| \psi \| \leq \limsup_{n \to \infty} \| \psi_n \| + \limsup_{n \to \infty} \| \psi_n - \psi \| = \infty \Rightarrow \psi \in \mathcal{S}(M_V) \]

spectrum:
inverse of \[ M_V : \left( M_V - \lambda \right)^{-1} \Phi = \left( \frac{1}{V - \lambda} \psi \right)(x) \]
\[ \lambda \notin \text{ ess ran } V \Rightarrow \exists \epsilon > 0 : \text{ dy-a.e. } x \in \mathfrak{m}, \quad V(x) - \lambda \geq \epsilon \]
\[ \Rightarrow \text{ ess ran } V = V = \text{ ess ran } V \]
\[ \text{ ess ran } V < \text{ ess ran } V \]
\[ \left( M_V - \lambda \right)^{-1} \text{ is bounded } \Rightarrow \text{ ess ran } \frac{1}{V(x) - \lambda} < \infty \]

opposite inclusion:
\[ \lambda \in \text{ ess ran } V \Rightarrow \forall \epsilon > 0 \quad \mu \left( \left\{ x \mid |V(x) - \lambda| < \epsilon \right\} \right) > 0 \]
\[ \mu \left( \left\{ x \mid \frac{1}{V(x) - \lambda} < \frac{1}{\epsilon} \right\} \right) > 0 \]
\[ \Rightarrow \text{ ess ran } \frac{1}{V(x) - \lambda} = \infty \Rightarrow \left( M_V - \lambda \right)^{-1} \text{ is not bounded} \]
Exercise 2) alternative full proof of \( \text{Ker}(\lambda - A^*) = \text{Ker}(\lambda - A) \)

\( \lambda \in \mathbb{C}, \quad \|m\| = 1, \quad v, w \in \mathbb{R}^n \)

\( \lambda \text{ symmetric} \Rightarrow \| (\lambda - A) v \| \geq \| m \| \quad \forall v \in \text{Ker}(\lambda - A) \) \hspace{1cm} (1)

\( \lambda \text{ closed} \Rightarrow \text{Ker}(\lambda - A) \) is closed subspace of \( \mathbb{R}^n \)

\( \text{Ker}(\lambda - A^*) = \text{Ker}(\lambda - A)^\perp \) \hspace{1cm} (2)

We want to show that \( \dim \text{Ker}(\lambda - A^*) \) is locally constant in \( \mathbb{C} \).

If \( \eta \in \mathbb{C} \):

\( \dim \text{Ker}(\lambda + \eta - A^*) = \dim \text{Ker}(\lambda - A^*) \)

\( \|m\| = 1 \), \( m \in \text{Ker}(\lambda + \eta - A^*) \)

we suppose that \( (x, y) = 0 \), \( x \in \text{Ker}(\lambda - A^*) \) and show contradiction.

using (2) \( m \in \text{Ker}(\lambda - A) \) \( \Rightarrow 0 \in \text{Ker}(\lambda - A) \) \( \Rightarrow 0 = \lambda \).

This is a contradiction for \( \|m\| = 1 \) because (1) implies \( \|m\| = 1 \) if \( \lambda \in \mathbb{C} \).

We can conclude that for \( |\eta| < \lambda \) there is \( m \in \text{Ker}(\lambda + \eta - A^*) \)

and is in \( \text{Ker}(\lambda - A^*)^\perp \). This implies that \( \dim \text{Ker}(\lambda + \eta - A^*) = \dim \text{Ker}(\lambda - A^*) \)

We can use the same argumentation for \( |\eta| < \frac{|\lambda|}{2} \) to show that

\( \dim \text{Ker}(\lambda - A^*) = \dim \text{Ker}(\lambda + \eta - A^*) \).

Putting together we have \( \dim \text{Ker}(\lambda - A^*) = \dim \text{Ker}(\lambda + \eta - A^*) \) for \( |\eta| < \frac{|\lambda|}{2} \).

By this we proved that \( \dim \text{Ker}(\lambda - A^*) \) is locally constant. Because we had no special conditions on \( \lambda \) we can conclude that the dimension is constant on upper half-plane of \( \mathbb{C} \) and on lower half-plane of \( \mathbb{C} \).

We note that these dimensions might not be equal.