Worksheet 3

Problem 1

a) 1) find adjoint of T_0

$$-(\phi,\psi'') = -[\bar{\phi}\psi']_0^\infty + (\phi',\psi') = -\underbrace{[\bar{\phi}\psi']_0^\infty}_{=0} + \underbrace{[\bar{\phi}'\psi]_0^\infty}_{=0} - (\psi'',\phi)$$

$T_0^*\psi = -\psi''$

$\mathcal{D}(T_0^*) = AC^2(\mathbb{R}^+)$

2) find deficiency subspaces

$-\psi'' = k^2\psi \qquad \psi_+ = e^{ikx} \qquad \psi_- = e^{-ikx}$

$k^2 = \pm i$

$k = \sqrt{\pm i} = \frac{\pm 1 + i}{\sqrt{2}}$

$\mathcal{K}_+$:

$k^2 = i$

$\psi_{++} = \exp\left(i\frac{1+i}{\sqrt{2}}x\right) \in L^2$

$\psi_{+-} = \exp\left(-i\frac{1+i}{\sqrt{2}}x\right) \notin L^2$

$n_+ = 1$

$\mathcal{K}_-$:

$k^2 = -i$

$\psi_{-+} = \exp\left(i\frac{-1+i}{\sqrt{2}}x\right) \in L^2$

$\psi_{--} = \exp\left(-i\,\frac{-1+i}{\sqrt{2}}x\right) \notin L^2$

$n_- = 1$

3) isometry: $U\psi_{++} = e^{i\theta}\psi_{-+}$

4) rewrite domain in nicer form $\psi(x) = \psi_0(x) + c\,\psi_{++} + c\,e^{i\theta}\psi_{-+} \quad \psi_0 \in \mathcal{D}(T_0)$

$$\psi'(0_+) = c\left(\frac{i-1}{\sqrt{2}} + \left(\frac{-1-i}{2}\right)e^{i\theta}\right) = \alpha c(1+e^{i\theta}) = \alpha\,\psi(0_+)$$

$(1+e^{i\theta})\alpha = e^{\frac{3i\pi}{4}} + e^{\frac{5i\pi}{4}}e^{i\theta}$

$\theta \neq \pi \qquad \alpha = -\frac{\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}{\cos\frac{\theta}{2}}$

self-adjoint extension: $T = -\frac{d^2}{dx^2} \quad \mathcal{D}(T) = \{\psi \in AC^2(\mathbb{R}^+) \mid \alpha\,\psi(0_+) = \psi'(0_+)\}$

b) 1) find adjoint of T_0

in the same way as a)

$$T_0^* = -\frac{d^2}{dx^2}$$

$$\mathcal{D}(T_0^*) = \{ f \in AC^2((0,1)) \}$$

2) defining subspaces - we use solution of a)

K_+:

$$\psi_{++} = \exp(i\sqrt{i}\,x) \in L^2(0,1) \Rightarrow n_+ = 2$$
$$\psi_{+-} = \exp(-i\sqrt{i}\,x) \in L^2(0,1)$$

K_-:

$$\psi_{-+} = \exp(i\sqrt{i}\,x) \in L^2((0,1)) \Rightarrow n_- = 2$$
$$\psi_{--} = \exp(-i\sqrt{i}\,x)$$

3) p. isometry U: $\quad U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$

~~$U\psi = \ldots$~~

$$U = |\psi_{-+}\rangle U_{11} \langle \psi_{++}| + |\psi_{--}\rangle U_{21} \langle \psi_{++}| + |\psi_{-+}\rangle U_{12} \langle \psi_{+-}| + |\psi_{--}\rangle U_{22} \langle \psi_{+-}|$$

U is isometry $\Leftrightarrow \|Ux\|_y = \|x\|_X \Rightarrow |U_{11}|^2 + |U_{21}|^2 = 1$

$U: X \to Y$ $\qquad |U_{12}|^2 + |U_{22}|^2 = 1$

$$(x,y)_X = 0 = (Ux, Uy)_Y \qquad \overline{U_{11}}\,U_{12} + \overline{U_{21}}\,U_{22} = 0$$

$$U^* U = 1 \Rightarrow U^{-1} = U^* \Rightarrow |\det U| = 1 \qquad |U_{11}U_{22} - U_{12}U_{21}| = 1$$

Problem 2)

a) $p = i\frac{d}{dx}$

$\mathcal{D}(p) = \{ f \in AC^1(\mathbb{R}^+) \mid f(0) = 0 \}$

$f \in AC^1(\mathbb{R}^+) \Leftrightarrow f \in L^2(\mathbb{R}^+)$, f absolutely continuous, $f' \in L^2(\mathbb{R}^+)$

1) adjoint: p^*

$$(\psi, p\phi) = (\psi, i\phi') = i\underbrace{[\psi\phi]_0^\infty}_{\phi\in\mathcal{D}(p) \Rightarrow 0} - i(\psi', \phi) = (i\psi', \phi) = (p\psi, \phi)$$

$p^* = i\frac{d}{dx}$

$\mathcal{D}(p^*) = AC^1(\mathbb{R}^+)$

2) deficiency subspaces

$i\psi' = k\psi \Rightarrow \psi_+' = \psi_+ \qquad \psi_+ = \exp x \notin L^2(\mathbb{R}) \qquad n_+ = 0 \Rightarrow p$ maximal

$\qquad\qquad\;\; \psi_-' = -\psi_- \qquad \psi_- = \exp(-x) \in L^2(\mathbb{R}) \qquad n_- = 1 \Rightarrow p$ non s.a.

b) $p = i\frac{d}{dx}$

$\mathcal{D}(p) = \{ \psi \in AC^1(\mathbb{R}^+) \mid f(0) = 0 = f(1) \}$

$$(\psi, p\phi) = (\psi, i\phi') = i\underbrace{[\psi\phi]_0^1}_{0} + i\underbrace{[\psi\phi]_1^\infty}_{0} - i(\psi', \phi) = (p\psi, \phi)$$

$\phi \in \mathcal{D}(p)$

$p^* = i\frac{d}{dx}$

$\mathcal{D}(p^*) = AC^1((0,1)) \oplus AC^1((1,\infty))$

2) deficiency subspaces

K_+: $\psi_+' = \psi_+ \qquad \psi_+ = \exp x \notin L^2((1,\infty))$, $\in L^2((0,1)) \Rightarrow n_+ = 1 \qquad \psi_{+1} = \exp x \quad x \in (0,1)$; $= 0 \quad x \in (1,\infty)$

K_-: $\psi_-' = -\psi_- \qquad \psi_- = \exp(-x) \in L^2((1,\infty))$, $\in L^2((0,1)) \Rightarrow n_- = 2 \qquad \psi_{-1} = \exp(-x) \quad x \in (0,1)$; $= 0 \quad x \in (1,\infty)$

$\psi_{-2} = \exp(-x) \quad x \in (1,\infty)$; $= 0 \quad x \in (0,1)$