Worksheet 4)

Exercise 1)

a) $\|V\psi\|^2 \le \tilde{a}^2\|T\psi\|^2 + \tilde{b}^2\|\psi\|^2 \le \tilde{a}^2\|T\psi\|^2 + \tilde{b}^2\|\psi\|^2 + 2\tilde{a}\tilde{b}\|T\psi\|\|\psi\| \qquad \tilde{a},\tilde{b}\ge 0$

$\le (\tilde{a}\|T\psi\| + \tilde{b}\|\psi\|)^2$

$\Downarrow$

$\|V\psi\| \le \tilde{a}\|T\psi\| + \tilde{b}\|\psi\|$

b) $\|V\psi\| \le a\|T\psi\| + b\|\psi\|$

$\|V\psi\|^2 \le a^2\|T\psi\|^2 + b^2\|\psi\|^2 + 2ab\|T\psi\|\,\|\psi\|$

$0 \le (-A\varepsilon + \frac{B}{\varepsilon})^2 \Rightarrow 2AB \le A^2\varepsilon^2 + \frac{B^2}{\varepsilon^2} \Rightarrow 2ab\|T\psi\|\,\|\psi\| \le a^2\tilde{\varepsilon}\|T\psi\|^2 + \frac{b^2\|\psi\|^2}{\tilde{\varepsilon}}$

$\|V\psi\|^2 \le a^2(1+\tilde{\varepsilon})\|T\psi\|^2 + b^2(1+\frac{1}{\tilde{\varepsilon}})\|\psi\|^2$

Exercise 2)

a) $(-\Delta+E)^{-1} = \int_0^\infty e^{-tE} e^{-t\Delta}\,dt = \begin{bmatrix} t = |x-y|^2 s \\ dt = |x-y|^2 ds \end{bmatrix} = (4\pi)^{-\frac{d}{2}} \int_0^\infty \exp\left(-\frac{1}{4s} - sE|x-y|^2\right) s^{-\frac{d}{2}} |x-y|^{-d+2}\,ds$

$= \begin{bmatrix} s = u^{-2} \\ ds = -2u^{-3}\,du \end{bmatrix} = (4\pi)^{-\frac{d}{2}} |x-y|^{-d+2}\, 2\int_0^\infty \exp\left(-\frac{u^2}{4} - \frac{E|x-y|^2}{u^2}\right) u^{d-3}\,du$

$= (4\pi)^{-\frac{d}{2}} |x-y|^{-d+2}\, 2\int_0^\infty \exp\left(-u^2(\tfrac{1}{4}-k) - \sqrt{k}\sqrt{E}|x-y| \underbrace{\left(\frac{\sqrt{k}\,u^2}{\sqrt{E}|x-y|} + \frac{\sqrt{E}|x-y|}{\sqrt{k}\,u^2}\right)}_{\ge 2}\right) u^{d-3}\,du$

$\le (4\pi)^{-\frac{d}{2}} |x-y|^{-d+2}\, 2\exp\left(-2\sqrt{k}\sqrt{E}|x-y|\right) \int_0^\infty \exp\left(-u^2(\tfrac{1}{4}-k)\right) u^{d-3}\,du$

$\le (4\pi)^{-\frac{d}{2}} |x-y|^{-d+2}\, 2\exp\left(-2\sqrt{k}\sqrt{E}|x-y|\right) I_{d-3}(\tfrac{1}{4}-k)$

Gauss integral $I_{d-3}(\frac{1}{4}-k) < \infty$

if $\frac{1}{4}-k > 0$

$I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$I_1(a) = \frac{1}{2a}$

$-I_n(a) = \frac{\partial}{\partial a} I_{n-2}(a)$

b) $|(V(-\Delta+E)^{-1}\varphi)(x)| = |V(x)|\left|\int_{\mathbb{R}^3} (-\Delta+E)^{-1}(x,y)\varphi(y)\,dy\right|$

$\le |V(x)| \int_{\mathbb{R}^3} (-\Delta+E)^{-1}(x,y)\,|\varphi(y)|\,dy$

$\le |V(x)| \left(\int_{\mathbb{R}^3} [(-\Delta+E)^{-1}(x,y)]^2\,dy\right)^{\frac{1}{2}} \|\varphi\|_2$

$\int_{\mathbb{R}^3} |(-\Delta+E)^{-1}(x,y)|^2\,dy \le \int_{\mathbb{R}^3} c_3^2 |x-y|^{-2} \exp(-4\sqrt{E}|x-y|)\,dy = c_3^2 4\pi \int_{\mathbb{R}^+} \exp(-4\sqrt{E}r)\,dr = c_3^2 4\pi \frac{1}{4\sqrt{E}}$

$\le c_3^2 \frac{\pi}{\sqrt{E}}$

Side note: $\|u * v\|_r = \|u\|_p \|v\|_q$

$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$

Young inequality

$\|V(-\Delta+E)^{-1}\varphi\|_2 \le \|V\|_2\, c_3^2 \frac{\pi}{\sqrt{E}} \|\varphi\|_2 \xrightarrow[E\to\infty]{} 0$

$\|V(-\Delta+E)^{-1}\varphi\|_2 \le \|V\|_2 \|(-\Delta+E)^{-1}\varphi\|_\infty \le \|V\|_2 \|(-\Delta+E^{-1})\|_2 \|\varphi\|_2$

$$\Rightarrow \|V(-\Delta+E)^{-1}\psi\| \leq C_E\|\psi\|$$

$$\psi = (-\Delta+E)\varphi$$

$$\|V\varphi\| \leq C_E\left(\|-\Delta\varphi+E\varphi\|\right) \leq C_E\|-\Delta\varphi\|+C_E E\|\varphi\|$$