Worksheet 5

Problem 1)

$$\|U(\lambda)\| = \left\|\sum_{n=0}^{\infty} \frac{(-i\lambda A)^n}{n!}\right\| \le \sum_{n=0}^{\infty} \frac{\|(-i\lambda A)^n\|}{n!} \le \sum_{n=0}^{\infty} \frac{\lambda^n \|A\|^n}{n!} = e^{\lambda\|A\|} < \infty$$

a) $$\|\exp(-i\lambda A) - 1\| = \left\|\sum_{n=1}^{\infty} \frac{(-i\lambda A)^n}{n!}\right\| \le \sum_{n=1}^{\infty} \frac{\lambda^n \|A\|^n}{n!} = e^{\lambda\|A\|} - 1 \xrightarrow[\lambda\to 0]{} 0$$

b) $$\left\|\frac{\exp(-i\lambda A) - 1}{\lambda} + iA\right\| = \left\|\frac{\sum_{n=2}^{\infty} \frac{(-i\lambda A)^n}{n!}}{\lambda}\right\| \le \frac{\sum_{n=2}^{\infty} \frac{\lambda^n \|A\|^n}{n!}}{\lambda} = \frac{\exp(\lambda\|A\|) - 1 - \|A\|\lambda}{\lambda}$$

$$\lim_{\lambda\to 0} \frac{\exp(\lambda\|A\|) - 1 - \|A\|\lambda}{\lambda} \underset{\text{L'Hopital}}{=} \lim_{\lambda\to 0} \frac{\|A\|\exp(\lambda\|A\|) - \|A\|}{1} = 0$$

c) $$U(\lambda)U(s) = \sum_{n=0}^{\infty} \frac{(-i\lambda A)^n}{n!} \sum_{k=0}^{\infty} \frac{(-isA)^k}{k!} = \sum_{n,k=0}^{\infty} \frac{(-i\lambda A)^n(-isA)^k}{n!\,k!} = \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-i\lambda A)^m(-isA)^{l-m}}{m!\,(l-m)!} =$$

$$= \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(-iA)^l \lambda^m s^{l-m}}{m!\,(l-m)!} = \sum_{l=0}^{\infty} \frac{(-iA)^l}{l!} \sum_{m=0}^{l} \frac{l!\,\lambda^m s^{l-m}}{m!\,(l-m)!} = \sum_{l=0}^{\infty} \frac{(-iA)^l(\lambda+s)^l}{l!} = U(\lambda+s)$$

d) $$\|U(\lambda+h) - U(\lambda)\| = \|U(\lambda)[U(h) - 1]\| \le \|U(\lambda)\|\,\|U(h) - 1\|$$

$$\lim_{h\to 0}\|U(\lambda+h) - U(\lambda)\| \le \|U(\lambda)\|\left[\lim_{h\to 0}\|U(h) - 1\|\right] \xrightarrow[\text{from a)}]{} 0$$

e) $$\left\|\frac{U(\lambda+h) - U(\lambda)}{h} - iU(\lambda)A\right\| \le \|U(\lambda)\|\left\|\frac{U(h) - 1}{h} - iA\right\|$$

$$\lim_{h\to 0}\left\|\frac{U(\lambda+h) - U(\lambda)}{h} - iU(\lambda)A\right\| \le \|U(\lambda)\|\lim_{h\to 0}\left\|\frac{U(h) - 1}{h} - iA\right\| \xrightarrow[\text{from b)}]{} 0$$

This implies $\lim_{h\to 0} \frac{U(\lambda+h) - U(\lambda)}{h} = -iU(\lambda)A$

$$U(\lambda)A = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n A^n}{n!} A = \sum_{n=0}^{\infty} \frac{(-i\lambda)^n A^{n+1}}{n!} = A\sum_{n=0}^{\infty} \frac{(-i\lambda A)^n}{n!} = AU(\lambda)$$

f) $$(\varphi, U(-\lambda)\psi) = \left(\varphi, \sum_{n=0}^{\infty} \frac{(i\lambda A)^n}{n!}\psi\right) = \sum_{n=0}^{\infty}\left(\varphi, \frac{(i\lambda A)^n}{n!}\psi\right) = \sum_{n=0}^{\infty}\left(\frac{(-i)^n\lambda^n A^n}{n!}\varphi, \psi\right) =$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-i\lambda A)^n}{n!}\varphi, \psi\right) = (U(\lambda)\varphi, \psi)$$

Problem 2)

$$S_N = \exp\left(\frac{A+B}{N}\right) = 1 + \frac{A+B}{N} + \frac{A^2+AB+BA+B^2}{2N^2} + \mathcal{O}(N^{-3})$$

$$T_N = \exp\left(\frac{A}{N}\right)\exp\left(\frac{B}{N}\right) = \left(1+\frac{A}{N}+\frac{A^2}{2N^2}+\mathcal{O}(N^{-3})\right)\left(1+\frac{B}{N}+\frac{B^2}{2N^2}+\mathcal{O}(N^{-3})\right) = 1+\frac{A+B}{N}+\frac{A^2+2AB+B^2}{2N^2}+\mathcal{O}(N^{-3})$$

$$S_N - T_N = \frac{BA-AB}{2N^2} + \mathcal{O}(N^{-3})$$

$$\|S_N - T_N\| \leq \frac{c}{N^2}$$

$$S_N^N - T_N^N = \sum_{i=0}^{N-1} S_N^i (S_N - T_N) T_N^{N-1-i}$$

$$\|S_N^N - T_N^N\| \leq N \cdot \|S_N - T_N\| \cdot \left(\max\{\|S_N\|, \|T_N\|\}\right)^N \leq \frac{c}{N}\exp(\|A\|+\|B\|)$$

$$\lim_{N\to\infty}\left(\exp\left(\frac{A}{N}\right)\exp\left(\frac{B}{N}\right)\right)^N = \exp(A+B)$$