Exercise 6

Problem 1

a) we use approximating operator $e_{\varepsilon}^{it\Delta} = \mathcal{F}^{-1}\circ M_{V(k,\varepsilon)}\circ\mathcal{F}$, $\varepsilon>0$; $V(k,\varepsilon)=\exp(-i|k|^2 t-\varepsilon|k|^2)$

$\psi\in L^1\cap L^2 \Rightarrow \hat\psi\in L^2$, $\hat\psi(x)<\infty\ \forall x$

$e^{-\varepsilon|y|^2}\hat\psi\in L^1 \Leftarrow e^{-\varepsilon|y|^2}\in L^2,\ \hat\psi\in L^2$; $\|\varphi\psi\|_{L^1}\le\|\varphi\|_{L^p}\|\psi\|_{L^q}$ $\frac{1}{p}+\frac{1}{q}=1$ $p,q>1$

$\|\exp(-\varepsilon|y|^2)\hat\psi\|_{L^2}\le\|\hat\psi\|_{L^2} \Rightarrow \exp(-\varepsilon|y|^2)\hat\psi\in L^1\cap L^2 \Rightarrow (e_\varepsilon^{it\Delta}\psi)(x)<\infty\ \forall x$

$$(e_\varepsilon^{it\Delta}\psi)(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}\exp(iyx)\exp(-it|y|^2-\varepsilon|y|^2)\int_{\mathbb{R}^n}\exp(-iyr)\psi(r)\,dr=$$

$$=(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(iy(x-r)-it|y|^2-\varepsilon|y|^2)\psi(r)\,dy\,dr=$$

$$=(2\pi)^{-n}\int_{\mathbb{R}^n}\psi(r)\left[\int_{\mathbb{R}^n}\exp(iy(x-r)-it|y|^2-\varepsilon|y|^2)\,dy\right]dr=*$$

$$-y_j(iy_jt-i(x_j-r_j)+\varepsilon y_j)=-(\varepsilon+it)\left(y_j-\frac{i(x_j-r_j)}{2(\varepsilon+it)}\right)^2-\frac{(x_j-r_j)^2}{4(\varepsilon+it)}$$

$$\int_{\mathbb{R}}\exp\left[-(\varepsilon+it)\left(y_j-\frac{i(x_j-r_j)}{2(\varepsilon+it)}\right)^2\right]dy_j=\sqrt{\frac{\pi}{\varepsilon+it}}$$

$$*=(2\pi)^{-n}\left(\frac{\pi}{\varepsilon+it}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\psi(r)\exp\left(-\frac{|x-r|^2}{4(\varepsilon+it)}\right)dr\xrightarrow[\varepsilon\to0]{}\left(\frac{1}{4\pi it}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\psi(r)\exp\left(-\frac{|x-r|^2}{4it}\right)dr$$

$$(i)^{-\frac{n}{2}}=\left(\exp\left(i\frac{\pi}{2}\right)\right)^{-\frac{n}{2}}=\exp\left(-i\frac{\pi n}{4}\right)$$

b) we show that the limit holds for a dense set $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and use the result of a). The fact that it is sufficient follows from boundness of $e^{it\Delta}$

Lemma: $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ dense in $L^2(\mathbb{R}^n)$

$\varphi_\varepsilon\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$

$\varphi_\varepsilon(x)=\underbrace{e^{-\varepsilon x^2}}_{L^2}\underbrace{\varphi(x)}_{L^2} \Rightarrow \varphi_\varepsilon\in L^1$

$\|\varphi-\varphi_\varepsilon\|_2^2=\int(1-e^{-\varepsilon|x|^2})^2|\varphi(x)|^2dx\xrightarrow[\varepsilon\to0]{}0$ we can switch lim and $\int$ because integral is uniformly bounded by $\|\varphi\|_2^2$

$$\left\| e^{is\Delta}\psi - \exp\left(-i\frac{\pi}{4}n\right)\int_{\mathbb{R}^n} \frac{\exp\left(i\frac{|x-y|^2}{4s} - \varepsilon|y|^2\right)}{(4\pi s)^{\frac{n}{2}}}\psi(y)\,dy \right\|_{L^2} = *$$

$$(T_s^{\varepsilon}\psi)(y) = \exp(is\Delta)\left[\exp(-\varepsilon x^2)\psi(x)\right](y)$$

$$* = \left\|\left(T_s^0 - T_s^{\varepsilon}\right)\psi\right\|_{L^2} \leq \|\exp(is\Delta)\|\,\|(1-\exp(-\varepsilon|x|^2))\psi\|_{L^2} \leq$$

$$\leq \|\exp is\Delta\|\,\|1-\exp(-\varepsilon|x|^2)\|_{\infty}\,\|\psi\|_{L^2} \xrightarrow[\varepsilon\to 0]{} 0$$

c) $i\partial_s \phi(s,x) \overset{?}{=} -\Delta_x \phi(s,x)$

$$i\partial_s\left(e^{is\Delta}\psi(x)\right) \overset{?}{=} -\Delta_x\left(e^{is\Delta}\psi(x)\right)$$

$$i\,\partial_s\left(\exp(is\Delta)\psi(x)\right) = i\exp\left(-i\frac{\pi}{4}n\right)\int_{\mathbb{R}^n}\left[\underbrace{\frac{-\frac{n}{2}\exp\left(i\frac{|x-y|^2}{4s}\right)}{(4\pi)^{\frac{n}{2}}\,s^{\frac{n}{2}+1}}} - \frac{i\frac{|x-y|^2}{4s^2}\exp\left(i\frac{|x-y|^2}{4s}\right)}{(4\pi)^{\frac{n}{2}}\,s^{\frac{n}{2}}}\right]\psi(y)\,dy$$

$$\frac{\partial}{\partial x_j\partial x_j}\left(f((x_i-y_i)(x_i-y_i))\right) = \frac{\partial}{\partial x_j}\left[\left(f'((x_i-y_i)(x_i-y_i)\right)2(x_j-y_j)\right] =$$

$$= 2nf'((x_i-y_i)(x_i-y_i)) + 4f''((x_i-y_i)(x_i-y_i))(x_j-y_j)(x_j-y_j)$$

$$\Downarrow$$

$$\Delta_x\left(\exp\left(i\frac{|x-y|^2}{4s}\right)\right) = 2n\frac{i}{4s}\exp\left(i\frac{|x-y|^2}{4s}\right) + 4\left(\frac{i}{4s}\right)^2\exp\left(i\frac{|x-y|^2}{4s}\right)|x-y|^2$$

$$-\Delta_x\left(e^{is\Delta}\psi(x)\right) = -\exp\left(-i\frac{\pi}{4}n\right)\int_{\mathbb{R}^n}\frac{\psi(y)}{(4\pi s)^{\frac{n}{2}}}\left(\underbrace{2n\frac{i}{4s}\exp\left(i\frac{|x-y|^2}{4s}\right)} + 4\left(\frac{i}{4s}\right)^2\exp\left(i\frac{|x-y|^2}{4s}\right)|x-y|^2\right)dy$$

$$i\partial_s\phi(s,x) = -\Delta_x\phi(s,x)$$

Problem 2

a) $(U_\lambda \psi, U_\lambda \psi) = \int_{\mathbb{R}^n} \lambda^{\frac{n}{2}} \overline{\psi}(\lambda x)\, \lambda^{\frac{n}{2}} \psi(\lambda x)\, dx = \lambda^n \int_{\mathbb{R}^n} |\psi|^2(\lambda x)\, dx = \begin{bmatrix} \lambda x_i = y_i \\ \lambda dx_i = dy_i \\ \lambda^n dx = dy \end{bmatrix} =$

$= \int_{\mathbb{R}^n} |\psi|^2 dy = (\psi, \psi)$

$(U_\lambda \psi)(x) = \lambda^{\frac{n}{2}} \psi(\lambda x)$

$U_{\lambda^{-1}} (U_\lambda \psi)(x) = \lambda^{-\frac{n}{2}} \lambda^{\frac{n}{2}} \psi\left(\frac{\lambda x}{\lambda}\right) = \psi(x)$

$U_{\lambda^{-1}} U_\lambda = 1 = U_\lambda U_{\lambda^{-1}}$

b) $\psi \in H^s(\mathbb{R}^n) \Leftrightarrow \int_{\mathbb{R}^n} |\psi|^2 dx < \infty \,;\, \int_{\mathbb{R}^n} (1+|x|^2)^s |\psi|^2 dx < \infty$

$\|U_\lambda \psi\|^2_{H^s} = \int_{\mathbb{R}^n} (1+|x|^2)^s |\psi(\lambda x)|^2 \lambda^n dx = \int_{\mathbb{R}^n} (1+|\frac{x}{\lambda}|^2)^s |\psi(x)|^2 dx \leq (\max\{\lambda, \frac{1}{\lambda}\})^{2s} \|\psi\|^2_{H^s}$

$\Rightarrow \psi \in H^s \Rightarrow U_\lambda \psi \in H^s$

c) $(U_\lambda \Delta U_\lambda^{-1} \psi)(x) = (U_\lambda \Delta)\left(\frac{\psi(\frac{x}{\lambda})}{\lambda^{\frac{n}{2}}}\right) = U_\lambda \left(\frac{\Delta \psi(\frac{x}{\lambda})}{\lambda^{\frac{n}{2}+2}}\right) = \frac{1}{\lambda^2} \Delta \psi(x)$

$(U_\lambda V U_{\lambda^{-1}} \psi)(x) = (U_\lambda V)\left(\lambda^{-\frac{n}{2}} \psi\left(\frac{x}{\lambda}\right)\right) = U_\lambda \left(V(x)\, \lambda^{-\frac{n}{2}} \psi\left(\frac{x}{\lambda}\right)\right) = V(\lambda x)\, \psi(x)$

Problem 3

$H_0 \psi_0 = E_0 \psi_0 \,,\, V = r^k \,,\, T = -\Delta$

$\Updownarrow$

$(AT + BV)\psi_0 = E_0 \psi_0$

$U_\lambda (AT + BV) U_\lambda^{-1} U_\lambda \psi_0 = E_0 U_\lambda \psi_0$

$(U_\lambda A T U_\lambda^{-1} + U_\lambda B V U_\lambda^{-1}) U_\lambda \psi_0 = E_0 U_\lambda \psi_0$

$(A \lambda^{-2} T + B \lambda^k V) U_\lambda \psi_0 = E_0 U_\lambda \psi_0$

$\left(T + \frac{B}{A} \lambda^{k+2} V\right) U_\lambda \psi_0 = \frac{\lambda^2 E_0}{A} U_\lambda \psi_0$

$k \neq 2 : \; U_\lambda \psi_0 = \psi_1 \,;\, \frac{\lambda^2 E_0}{A} = E_1 \,;\, \lambda = \left(\frac{A}{B}\right)^{-k-2}$

$\lambda = \left(\frac{\hbar^2}{2mC}\right)^{-k-2}$