Exercise 7

Problem 1

$b \Leftarrow$ trivial

$a \Rightarrow b$: $\forall \mu \in \rho(H)$: $R(\mu_0) - R(\mu) = (\mu_0 - \mu) R(\mu_0) R(\mu)$

$$\underbrace{FR(\mu_0)}_{\text{compact operator}} - FR(\mu) = \underbrace{\underbrace{FR(\mu_0)}_{\text{compact op.}} \underbrace{\left[(\mu - \mu_0) R(\mu) \right]}_{\text{bounded op.}}}_{\text{compact op.}}$$

$$\Downarrow$$

$F(R(\mu))$ is compact operator

Problem 2

a) $$\int_{\mathbb{R}^n} |\nabla f|^2 dx = \int_{\mathbb{R}^n} |\nabla (g |x|_\varepsilon^{-\alpha})|^2 dx = \int_{\mathbb{R}^n} |\nabla g|^2 |x|_\varepsilon^{-2\alpha} dx - 2\mathrm{Re}\left(\int_{\mathbb{R}^n} \alpha \bar{g} |x|_\varepsilon^{-2\alpha-2} x \cdot \nabla g \, dx \right)$$
$$+ \alpha^2 \int |g|^2 x^2 |x|_\varepsilon^{-2\alpha-4} dx = *$$

$$\int_{\mathbb{R}^n} \bar{g} |x|_\varepsilon^{-2\alpha-2} x \cdot \nabla g \, dx = -\int_{\mathbb{R}^n} \nabla \cdot \left(\bar{g} |x|_\varepsilon^{-2\alpha-2} x \right) g \, dx = -\int_{\mathbb{R}^n} (\nabla \bar{g}) |x|_\varepsilon^{-2\alpha-2} x g \, dx$$
$$- \int_{\mathbb{R}^n} |g|^2 \left((-2\alpha-2) |x|_\varepsilon^{-2\alpha-3} \frac{x}{|x|_\varepsilon} \cdot x + n |x|_\varepsilon^{-2\alpha-2} \right) dx$$

$$\Rightarrow 2\mathrm{Re} \int \bar{g} |x|_\varepsilon^{-2\alpha-2} x \nabla g \, dx = -\int |g|^2 \left(|x|_\varepsilon^{-2\alpha} \left(n - 2(\alpha+1) \frac{x^2}{|x|_\varepsilon^2} \right) |x|_\varepsilon^{-2} \right) dx$$

$$* = \int_{\mathbb{R}^n} |\nabla g|^2 |x|_\varepsilon^{-2\alpha} dx + \int_{\mathbb{R}^n} |f|^2 \left(n\alpha - 2\alpha(\alpha+1) \frac{x^2}{|x|_\varepsilon^2} + \alpha^2 x^2 |x|_\varepsilon^{-2} \right) |x|_\varepsilon^{-2} dx =$$

$$= \int_{\mathbb{R}^n} |\nabla g|^2 |x|_\varepsilon^{-2\alpha} dx + \int_{\mathbb{R}^n} |f|^2 \left(\alpha n - \alpha^2 \frac{x^2}{|x|_\varepsilon^2} - 2\alpha \frac{x^2}{|x|_\varepsilon^2} \right) |x|_\varepsilon^{-2} dx$$

b) $$\int_{\mathbb{R}^n} |\nabla f|^2 \geq \int_{\mathbb{R}^n} |f|^2 \left(\alpha n - (\alpha^2 + 2\alpha) \frac{x^2}{|x|_\varepsilon^2} \right) |x|_\varepsilon^{-2} dx$$
$$\downarrow \varepsilon \to 0$$
$$\int_{\mathbb{R}^n} |\nabla f|^2 \geq \int_{\mathbb{R}^n} |f|^2 \left(-\alpha^2 + \alpha(n-2) \right) x^{-2} dx$$

optimal α) minimize $-\alpha^2 + \alpha(n-2) \Rightarrow -2\alpha + n - 2 = 0 \quad \alpha = \frac{2-n}{-2}$

$\Rightarrow$ minimal value of $-\alpha + \alpha(n-2) = \frac{(n-2)^2}{4}$

$$\int_{\mathbb{R}^n} |\nabla f|^2 \geq \left(\frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{|f|^2}{x^2} dx$$

Problem 3

$$\int_0^\infty \frac{|\psi(x)|^2}{x^2}\,dx = -\int_0^\infty \frac{d}{dx}\left(\frac{1}{x}\right)|\psi(x)|^2\,dx = \int_0^\infty \frac{1}{x}\,2\mathrm{Re}\{\overline{\psi(x)}\,\psi'(x)\}\,dx \leq 2\sqrt{\int_0^\infty \frac{|\psi(x)|^2}{x^2}dx}\sqrt{\int_0^\infty |\psi'(x)|^2 dx}$$

$$\int_0^\infty |\psi'(x)|^2\,dx \geq \frac{1}{4}\int_0^\infty \frac{|\psi(x)|^2}{x^2}\,dx \qquad \psi \in C_0^\infty(\mathbb{R}^+) \text{ it is sufficient, } C_0^\infty(\mathbb{R}^+) \text{ dense in } H_0^1(\mathbb{R}^+)$$

Problem 4

a) again sufficient to check for $\psi \in C_0^\infty(\mathbb{R}^+)$ and extend to $H_0^1(\mathbb{R}^+)$

$$a(\psi) := \int_0^\infty \left(|\psi'(x)|^2 - \frac{1}{4}\frac{|\psi(x)|^2}{x^2}\right)dx = \int_0^\infty \left|\frac{d}{dx}\frac{\psi(x)}{\sqrt{x}}\right|^2 x\,dx \geq 0$$

$$\left|\left(\frac{\psi(x)}{\sqrt{x}}\right)'\right|^2 = \left(\frac{\psi'}{x}\right)^2 + \frac{1}{4}\frac{\psi^2}{x^3} - \frac{\overline{\psi}'\psi + \psi'\overline{\psi}}{2x^2} \qquad \int_{\mathbb{R}^+} -\frac{\overline{\psi}'\psi}{x}\,dx = \int_{\mathbb{R}^+} \overline{\psi}\left(\frac{\psi'}{x} - \frac{\psi}{x^2}\right)dx \Rightarrow -\int_{\mathbb{R}^+}\frac{\psi'\overline{\psi}}{x}dx = \int_{\mathbb{R}^+}\frac{\psi^2}{x^2}dx$$

$$a(\psi) = 0 \Leftrightarrow \left(\frac{\psi(x)}{\sqrt{x}}\right)' = 0 \;\; \forall x \Rightarrow \psi(x) = C\sqrt{x} \notin H_0^1(\mathbb{R}^+) \text{ unless } C = 0$$

b) we construct a sequence of functions $\psi \rightsquigarrow \sqrt{x}$

$$\psi_\varepsilon(x) = \sqrt{2\varepsilon}\; x^{\left(\frac{1}{2}+\varepsilon\right)\mathrm{sgn}(1-x)}$$

$$\|\psi_\varepsilon(x)\| = \int_0^1 2\varepsilon\, x^{1+2\varepsilon}dx + \int_1^\infty 2\varepsilon\, x^{-1-2\varepsilon}dx = \left[2\varepsilon\frac{x^{2+2\varepsilon}}{2+2\varepsilon}\right]_0^1 + \left[x^{-2\varepsilon}\right]_1^\infty = \frac{\varepsilon}{1+\varepsilon} + 1$$

$$a(\psi_\varepsilon) = \int_0^1 2\varepsilon^3 x^{-1+2\varepsilon}dx + \int_1^\infty 2\varepsilon(1+\varepsilon)^2 x^{-3-2\varepsilon}dx = \varepsilon^2 + \frac{2\varepsilon(1+\varepsilon)^2}{2+2\varepsilon} = \varepsilon^2 + \varepsilon(1+\varepsilon) = \varepsilon(1+2\varepsilon)$$

$$\text{i.e.: } \varepsilon \to 0 \Rightarrow \|\psi_\varepsilon(x)\| \to 1 \quad a(\psi_\varepsilon) \to 0$$

$$\frac{\int_0^\infty |\psi_\varepsilon'(x)|^2\,dx}{\int_0^\infty \frac{|\psi_\varepsilon(x)|^2}{x^2}\,dx} = \frac{\int_0^1 2\varepsilon\left(\frac{1}{2}+\varepsilon\right)^2 x^{-1-2\varepsilon}dx + \int_1^\infty 2\varepsilon\left(-\frac{1}{2}-\varepsilon\right)^2 x^{-3-2\varepsilon}dx}{\int_0^1 2\varepsilon\, x^{-1-2\varepsilon}dx + \int_1^\infty 2\varepsilon\, x^{-3-2\varepsilon}dx} = \left(\frac{1}{2}+\varepsilon\right)^2 \xrightarrow{\varepsilon \to 0} \frac{1}{4}$$