Problem 1

a) $f, g \in L^2(\mathbb{R}^d) \Rightarrow f(x)g(p)$ Hilbert-Schmidt kernel $p = i\nabla$

$P = M_f \mathcal{F}^{-1} M_g \mathcal{F}$

$\operatorname{tr} P^* P = \operatorname{tr}\left(\mathcal{F}^{-1} M_{\bar g} \mathcal{F} M_{\bar f} M_f \mathcal{F}^{-1} M_g \mathcal{F}\right) = \operatorname{tr}\left(M_{\bar g} \mathcal{F} M_{\bar f} M_f \mathcal{F}^{-1} M_g\right)$

$$\left(M_f \mathcal{F}^{-1} M_g \varphi\right)(x) = \int_{\mathbb{R}^d} f(x) \frac{1}{(2\pi)^{\frac{d}{2}}} e^{ix\eta} g(\eta) \varphi_\eta \, d\eta$$

$$\left\| M_f \mathcal{F}^{-1} M_g \right\|_{HS}^2 = \iint_{\mathbb{R}^{2d}} \left| f(x) \frac{1}{(2\pi)^{\frac{d}{2}}} e^{ix\eta} g(\eta) \right|^2 d\eta \, dx = \frac{1}{(2\pi)^d} \|f\|_2^2 \|g\|_2^2 < \infty$$

$g(x) = (x^2 + i)^{-1}$

$\|g(x)\|_2 < \infty$ if $d \leq 3$

$\Downarrow$

P H-S $\Rightarrow P$ compact $\Rightarrow \forall V \in L^2(\mathbb{R}^d) : V(-\Delta + i)^{-1}$ compact

b) $p = 2 \quad n \leq 3$

$V = V_{1,\varepsilon} + V_{2,\varepsilon}$

$V_{1,\varepsilon} \in L^2, \quad V_{2,\varepsilon} \in L^\infty_\varepsilon \quad \|V_{2,\varepsilon}\|_\infty \leq \varepsilon$

$V \in L^2 \Rightarrow V(-\Delta + i)^{-1}$ compact

$K = V(-\Delta + i)^{-1} = V_{1,\varepsilon}(-\Delta + i)^{-1} + V_{2,\varepsilon}(-\Delta + i)^{-1}$

$\|K - V_{1,\varepsilon}(-\Delta + i)^{-1}\| = \|V_{2,\varepsilon}(-\Delta + i)^{-1}\| \leq \varepsilon$

$\Downarrow$

$$\lim_{\varepsilon \to 0} \underbrace{V_{1,\varepsilon}(-\Delta + i)^{-1}}_{\text{compact}} = \underbrace{K}_{\Rightarrow \text{ compact}}$$

$p > \frac{n}{2} \quad n \geq 4$

it is sufficient to check $V(-\Delta + i)^{-1}$ compact for $V \in L^p(\mathbb{R}^n)$ and the rest as above

we already have $f, g \in L^2(\mathbb{R}^n) \Rightarrow f(x)g(p)$ Hilbert-Schmidt op.

$V_1(x) = \chi_R V(x) \quad V(x) \in L^p(\mathbb{R}^n), \ \chi_R(x) = 1 \ |x| < R \quad \chi_R(x) = 0 \ |x| \geq R$

$\Downarrow$

$V_1(x) \in L^2$

$\frac{1}{p^2 + i} 1\!\!1_{|p| < L} \in L^2$

$\Downarrow$

$\chi_R V \frac{1}{p^2 + i} 1\!\!1_{|p| < L}$ is compact

$$V_1 \frac{1}{p^2+i} = V_1 \frac{1}{p^2+i}\left(1\!\!1_{|p|<L} + 1\!\!1_{|p|\geq L}\right)$$

$$\left\| V_1 \frac{1}{p^2+i}\left(1\!\!1 - 1\!\!1_{|p|<L}\right)\right\| \leq \left\| V_1 \frac{1}{p^2+i} 1\!\!1_{|p|\geq L}\right\|$$

$$\|V\varphi\|_2 \leq a\|p^2\varphi\|_2 + b\|\varphi\|_2 \qquad \varphi = \frac{1}{p^2+i} 1\!\!1_{|p|\geq L}\psi$$

$$\left\|V\frac{1}{p^2+1} 1\!\!1_{|p|\geq L}\psi\right\| \leq a\left\|\frac{p^2}{p^2+i} 1\!\!1_{|p|\geq L}\psi\right\| + b\left\|\frac{1}{p^2+i} 1\!\!1_{|p|\geq L}\psi\right\| \leq a\|\psi\| + b\frac{1}{L^2+1}\|\psi\|$$

$$\lim_{\substack{L\to\infty \\ \|\psi\|=1}} \left\|V\frac{1}{p^2+1} 1\!\!1_{|p|\geq L}\psi\right\| \leq a \quad \forall a>0$$

$$\Downarrow$$

$$\underbrace{V_1 \frac{1}{p^2+i} 1\!\!1_{|p|<L}}_{\text{compact}} \xrightarrow[n\to 0]{} \underbrace{V_1 \frac{1}{p^2+i}}_{\text{compact}}$$

compact $\Rightarrow$ compact

$$\left\|(V-V_1)\frac{1}{p^2+i}\varphi\right\|_2 \leq \|V-V_1\|_{L^r}\left\|\frac{1}{p^2+i}\varphi\right\|_{L^q} = *$$

$$\left\|\frac{1}{p^2+i}\varphi\right\|_q \leq a\left\|\frac{p^2}{p^2+i}\varphi\right\|_2 + b\left\|\frac{1}{p^2+i}\varphi\right\|_2 \underset{\|\varphi\|_2=1}{\leq} a+b$$

$$* \leq |a+b|\,\|V-V_1\|_{L^r} \xrightarrow[R\to\infty]{} 0$$

$$\Downarrow$$

$$\underbrace{\chi_R V\frac{1}{p^2+i} 1\!\!1_{|p|<L}}_{\text{compact}} \xrightarrow[\substack{R\to\infty\\ L\to\infty}]{} \underbrace{V\frac{1}{p^2+i}}_{\text{compact}}$$

compact $\Rightarrow$ in norm compact

Problem 2

$$\partial_{x_1} = \partial_{x_1} y_{CM}\, \partial_{y_{CM}} + \partial_{x_1} y\, \partial_y = \frac{m_1}{M}\partial_{y_{CM}} + \partial_y$$

$$\partial_{x_2} = \partial_{x_2} y_{CM}\, \partial_{y_{CM}} + \partial_{x_2} y\, \partial_y = \frac{m_2}{M}\partial_{y_{CM}} - \partial_y$$

generally transformation of coordinate is unitary operator from $L^2(\mathbb{R}^n, dx)$ to $L^2(M(\mathbb{R}^n), \sqrt{|J|}\, dy)$; $M(x) = y$

$$\partial_{x_1x_1} = \left(\frac{m_1}{M}\right)^2 \partial_{y_{CM}y_{CM}} + 2\frac{m_1}{M}\partial_{y y_{CM}} + \partial_{yy}$$

$$\partial_{x_2x_2} = \left(\frac{m_2}{M}\right)^2 \partial_{y_{CM}y_{CM}} - 2\frac{m_2}{M}\partial_{y y_{CM}} + \partial_{yy}$$

$$\frac{\partial_{x_1x_1}}{m_1} + \frac{\partial_{x_2x_2}}{m_2} = \frac{1}{M}\partial_{y_{CM}y_{CM}} + \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\partial_{yy}$$

$$\Downarrow$$

$$\frac{\Delta_1}{m_1} + \frac{\Delta_2}{m_2} \approx \frac{p_{CM}^2}{M} + \mu\, p_y^2 \qquad p_{CM} = -i\frac{\partial}{\partial y_{CM}} \qquad p_y = -i\frac{\partial}{\partial y}$$

$$U\psi(x_1, x_2) = \tilde{\psi}(y_{CM}, y) \qquad U: H^2(\mathbb{R}^6) \to H^2(\mathbb{R}^6, \sqrt{|J|}\, dy_{CM}\, dy)$$

$$J_{ij} = \frac{\partial x_i}{\partial y_j} \qquad \vec{x}_1 = \frac{\vec{y}_{CM} + \frac{m_2}{M}\vec{y}}{\frac{m_1}{M}+\frac{m_2}{M}} \qquad \vec{x}_2 = \frac{\vec{y}_{CM} - \frac{m_1}{M}\vec{y}}{\frac{m_1}{M}+\frac{m_2}{M}}$$

$$= \vec{y}_{CM} + \frac{m_2}{M}\vec{y} \qquad\qquad = \vec{y}_{CM} - \frac{m_1}{M}\vec{y}$$

$$|J| = \left|\begin{pmatrix} 1 & 0 & 0 & \frac{m_2}{M} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{m_2}{M} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{m_2}{M} \\ 1 & 0 & 0 & -\frac{m_1}{M} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{m_1}{M} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{m_1}{M} \end{pmatrix}\right| = \left|\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\right| = 1$$

Problem 3

group property: $U(\alpha)U(\beta) \stackrel{?}{=} U(\alpha+\beta)$

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix} = \begin{pmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$$

strong continuity: $\|(U(\lambda)-1)\psi\| \xrightarrow[\lambda\to 0]{} 0$

sufficient to check for a dense subset of $L^2(\mathbb{R}^2)$, i.e. $C_0^\infty(\mathbb{R}^2)$

$f(x) \in C_0^\infty(\mathbb{R}^2)$

$f(R(x)) - f(x) = \nabla f(\xi)(R-1)x$

$|f(R(x)) - f(x)| \le |\nabla f(\xi)|\,|R-1|\,|x| \qquad \sigma(R-1) = \{\cos\varphi - 1 \pm i\sin\varphi\} \xrightarrow[\varphi\to 0]{} 0$

$\Rightarrow f(Rx) \to f(x)$ pointwise

b) $\left.\frac{dU}{d\varphi}\right|_{\varphi=0} = iA \qquad U = e^{i\varphi A}$

$$\frac{d}{d\varphi}\left(U(\varphi) f(x)\right) = \frac{d}{d\varphi}\left(f(R(x))\right) = \nabla f(R(x)) \cdot \begin{pmatrix} -\sin\varphi\, x_1 + \cos\varphi\, x_2 \\ -\cos\varphi\, x_1 + \sin\varphi\, x_2 \end{pmatrix} = -i\left(p_1, p_2\right)\left(f(R(x))\right) \cdot \begin{pmatrix} -\sin\varphi\, x_1 + \cos\varphi\, x_2 \\ -\cos\varphi\, x_1 + \sin\varphi\, x_2 \end{pmatrix}$$

$$\left.\frac{d}{d\varphi}\left(U(\varphi) f(x)\right)\right|_{\varphi=0} = -i\left(x_2 p_1 - x_1 p_2\right) f(x) = iL f(x)$$