Exercise 9

Problem 1

a) $f \in C_0^\infty$

$$[H_0, \varsigma_R] f = -\Delta(\varsigma_R f) + \varsigma_R \Delta f = -(\Delta \varsigma_R) f - (\nabla \varsigma_R) \nabla f$$

$$\nabla \varsigma_R(x) = \frac{1}{R} (\nabla \varsigma)\left(\frac{x}{R}\right)$$

$$\Delta \varsigma_R(x) = \frac{1}{R^2} (\Delta \varsigma)\left(\frac{x}{R}\right)$$

$$\| [H_0, \varsigma_R] (H_0+1)^{-1} \| \leq 2 \frac{1}{R} \left\| (\nabla \varsigma)\left(\frac{x}{R}\right) \nabla (H_0+1)^{-1} \right\| + \frac{1}{R^2} \left\| (\Delta\varsigma)\left(\frac{x}{R}\right) (H_0+1)^{-1} \right\| \leq$$

$$\leq 2 \frac{1}{R} \underbrace{\|\nabla \varsigma\|_\infty}_{<\infty} \underbrace{\left\| \frac{p}{p^2+1} \right\|}_{<1} + \frac{1}{R^2} \underbrace{\|\Delta \varsigma\|_\infty}_{<\infty} \underbrace{\|(H_0+1)^{-1}\|}_{<\infty}$$

b) $[H_0+V, \varsigma_R] = [H_0, \varsigma_R]$

$$\| [H_0+V, \varsigma_R] (H_0+V+i)^{-1} \| \leq \underbrace{\| [H_0, \chi_R] (H_0+i)^{-1} \|}_{\to 0 \text{ from a)}} \| (H_0+i)(H_0+V+i)^{-1} \|$$

$$\left\| \frac{H_0+i}{H_0+V+i} \right\| = \left\| 1 - \frac{V}{H_0+V+i} \right\| \leq 1 + \left\| \frac{V}{H_0+V+i} \right\|$$

$\|Vx\| \leq a\|Hx\| + c\|x\| \leq a\|(H+i)x\| + b\|x\|$

$-b\|x\| + (1-a)\|(H+i)x\| \leq \|(H+i)x\| - \|Vx\| \leq$

$\leq \|(H+i+V)x\| \Rightarrow \|(H+i)x\| \leq \frac{1}{1-a}\|(H+i+V)x\| + \frac{b}{1-a}\|x\|$

$x = (H+i+V)^{-1} y \Rightarrow \|(H+i)(H+i+V)^{-1} y\| \leq \frac{1}{1-a}\|y\| + \frac{b}{1-a}\|(H+i+V)^{-1} y\|$

Problem 2

a) $\Sigma(H) = \sigma_{ess}(H) \Rightarrow \exists u_n$ Weyl sequence $\|(H-\lambda_0) u_n\| \to 0$

$$\forall \phi \quad \Sigma_R(H) \leq (\phi, H\phi) \leq (u_n, H u_n) = \underbrace{(u_n, (H-\lambda_0) u_n)}_{\to 0} + \lambda_0 \underbrace{(u_n, u_n)}_{1} \Rightarrow \Sigma_R(H) \leq \lambda_0$$

b) $\forall R_n \ \exists \psi_{m_n}^{(n)}$ supp$(\psi_{m_n}) \in B_{R_n}(0)^c$, $\|\psi_{m_n}\| = 1$, $(\psi_{m_n}, H\psi_{m_n}) \to \Sigma_{R_n}(H)$

we choose m_n such that: $\Sigma_{R_n}(H) \leq (\psi_{m_n}, H\psi_{m_n}) \leq \Sigma_{R_n}(H) + \frac{1}{n}$

$\Rightarrow (\underbrace{\psi_{m_n}}_{\phi_n}, H\psi_{m_n}) \to \alpha$

c) ϕ_n weakly to 0 $\quad P_\varepsilon$ finite rank operator $\Rightarrow P_\varepsilon$ compact $\Rightarrow \lim_{n\to\infty} \|P_\varepsilon \phi_n\| = 0$

d) $(\phi, H P_\varepsilon^\perp \phi) = (\phi, H P_\varepsilon^{\perp 2} \phi) = (P_\varepsilon^\perp \phi, H P_\varepsilon^\perp \phi) \geq (\lambda_0 - \varepsilon) \| P_\varepsilon^\perp \phi \|^2$

e) $\underbrace{(\phi_n, H \phi_n)}_{\downarrow\, \alpha} \geq (\lambda_0 - \varepsilon) \underbrace{\| P_\varepsilon^\perp \phi_n \|^2}_{\downarrow\, 1} + c \underbrace{\| P_\varepsilon \phi_n \|^2}_{\downarrow\, 0}$

$$\Downarrow$$

$$\alpha \geq \lambda_0 - \varepsilon$$

f) $\lambda_0 \leq \alpha$ (a) $\quad \alpha \geq \lambda_0$ (e) $\quad \Rightarrow \alpha = \lambda_0$