

Exercise 9

Problem 1

a) $f \in C_0^\infty$

$$[H_0, S_R]f = -\Delta(S_R f) + S_R \Delta f = -(\Delta S_R)f - (\nabla S_R) \nabla f$$

$$\nabla S_R(x) = \frac{1}{R} (\nabla S) \left(\frac{x}{R} \right)$$

$$\Delta S_R(x) = \frac{1}{R^2} (\Delta S) \left(\frac{x}{R} \right)$$

$$\begin{aligned} \|[H_0, S_R](H_0 + 1)^{-1}\| &\leq 2 \frac{1}{R} \|(\nabla S) \left(\frac{x}{R} \right) \cdot \nabla (H_0 + 1)^{-1}\| + \frac{1}{R^2} \|(\Delta S) \left(\frac{x}{R} \right) (H_0 + 1)^{-1}\| \\ &\leq 2 \frac{1}{R} \underbrace{\|\nabla S\|_\infty}_{< \infty} \underbrace{\left\| \frac{1}{R^{i+1}} \right\|}_{< 1} + \frac{1}{R^2} \underbrace{\|\Delta S\|_\infty}_{< \infty} \underbrace{\|(H_0 + 1)^{-1}\|}_{< \infty} \end{aligned}$$

b) $[H_0 + V, S_R] = [H_0, S_R]$

$$\|[H_0 + V, S_R](H_0 + V + i)^{-1}\| \leq \underbrace{\|[H_0, S_R](H_0 + i)^{-1}\|}_{\rightarrow 0 \text{ from a)}} \|(H_0 + i)(H_0 + V + i)^{-1}\|$$

$$\left\| \frac{H_0 + i}{H_0 + V + i} \right\| = \left\| 1 - \frac{V}{H_0 + V + i} \right\| \leq 1 + \left\| \frac{V}{H_0 + V + i} \right\|$$

$$\begin{aligned} \|Vx\| &\leq a\|x\| + c\|x\| \leq a\|(H+i)x\| + \|Hx\| \\ -a\|x\| + (1-a)\|(H+i)x\| &\leq \|(H+i)x\| - \|Vx\| \leq \\ &\leq \|(H+i+V)x\| \Rightarrow \|(H+i)x\| \leq \frac{a}{1-a}\|(H+i+V)x\| + \frac{c}{1-a}\|x\| \\ x = (H+i+V)^{-1}y &\Rightarrow \|(H+i)(H+i+V)^{-1}y\| \leq \frac{1}{1-a}\|y\| + \frac{c}{1-a}\|(H+i+V)^{-1}y\| \end{aligned}$$

Problem 2

a) $Z(H) = G_{ess}(H) \Rightarrow \exists m_n$ which sequence $\|(H - \lambda_0) m_n\| \rightarrow 0$

$$\forall \phi \quad \sum_R(H) \leq (\phi, H \phi) \leq \underbrace{(m_n, H m_n)}_{\rightarrow 0} + \lambda_0 \underbrace{(m_n, m_n)}_1 \Rightarrow \sum_R(H) \leq \lambda_0$$

b) $\forall R_n \exists \psi_{m_n}$ with $\text{supp}(\psi_{m_n}) \in B_{R_n}(0)^c, \|\psi_{m_n}\| = 1, (\psi_{m_n}, H \psi_{m_n}) \rightarrow \sum_R(H)$

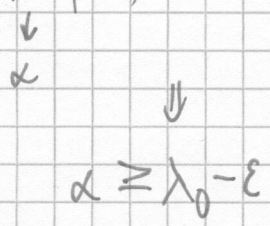
we choose m_n such that: $\sum_R(H) \leq (\psi_{m_n}, H \psi_{m_n}) \leq \sum_R(H) + \frac{1}{n}$

$$\Rightarrow \underbrace{(\psi_{m_n}, H \psi_{m_n})}_{\phi_n} \rightarrow \alpha$$

c) ϕ_n weakly to 0 P_ϵ finite rank operator $\Rightarrow P_\epsilon$ compact $\Rightarrow \lim_{n \rightarrow \infty} \|P_\epsilon \phi_n\| = 0$

$$d) (\phi, H P_{\varepsilon}^{\perp} \phi) = (\phi, H P_{\varepsilon}^{\perp 2} \phi) = (P_{\varepsilon}^{\perp} \phi, H P_{\varepsilon}^{\perp} \phi) \cong (\lambda_0 - \varepsilon) \|P_{\varepsilon}^{\perp} \phi\|^2$$

$$e) (\phi_m, H \phi_m) \cong (\lambda_0 - \varepsilon) \|P_{\varepsilon}^{\perp} \phi_m\|^2 + \|P_{\varepsilon} \phi_m\|^2$$



$$f) \lambda_0 \leq \alpha \quad (a) \quad \alpha \geq \lambda_0 \quad (e) \quad \Rightarrow \quad \alpha = \lambda_0$$

$$\|aH + b\| \geq \|a\| \|H\| \geq \|a\| \lambda_0$$

$$\Rightarrow \|a\| \geq \frac{\|aH + b\|}{\lambda_0}$$

$$\|a\| \geq \frac{\|a\| \lambda_0 + \|b\|}{\lambda_0} \Rightarrow \|a\| \geq \frac{\|b\|}{\lambda_0 - \lambda_0} = \infty$$

$$\frac{\|aH + b\|}{\|a\|} \geq \lambda_0 - \frac{\|b\|}{\|a\|}$$

$$\alpha \geq \lambda_0 \Rightarrow \sum_{k=1}^n (\alpha - \lambda_k) \langle \phi_k, \phi \rangle^2 \geq 0$$

$$\Rightarrow \sum_{k=1}^n \alpha \langle \phi_k, \phi \rangle^2 - \sum_{k=1}^n \lambda_k \langle \phi_k, \phi \rangle^2 \geq 0$$

$$\Rightarrow \alpha \sum_{k=1}^n \langle \phi_k, \phi \rangle^2 - \langle \phi, H \phi \rangle \geq 0$$

$$\Rightarrow \alpha \|\phi\|^2 - \langle \phi, H \phi \rangle \geq 0$$

$$\Rightarrow \alpha \geq \frac{\langle \phi, H \phi \rangle}{\|\phi\|^2} = \lambda_0$$

$$0 = \|\phi_0\|^2 = \langle \phi_0, \phi_0 \rangle = \langle \phi_0, H \phi_0 \rangle = \lambda_0 \|\phi_0\|^2$$