Institut für Analysis
Dr. Ioannis Anapolitanos
Dr. Nikolaos Pattakos

SS2020
23/04/2020

Mathematical Methods in Quantum Mechanics II

1st Exercise Sheet

Exercise 1:

Consider the Hamiltonian operator

$$H_{N,Z} = \sum_{i=1}^{N} \Big(-\frac{1}{2}\Delta_{x_i} - \frac{Z}{|x_i|}\Big) + \sum_{1\le i<j\le N} \frac{1}{|x_i - x_j|},$$

where $x_1, \dots, x_N \in \mathbb{R}^3$, and the space

$$L^2_a(\mathbb{R}^{3N}) = \Big\{ \psi \in L^2(\mathbb{R}^{3N}) : \psi(x_1, \dots, x_N) = (-1)^\sigma \psi(x_{\sigma(1)}, \dots, x_{\sigma(N)}), \text{for all } \sigma \in S_N \Big\}.$$

Show the following:

1. $(L^2_a(\mathbb{R}^{3N}), \|\cdot\|_2)$ is a Hilbert space. (Hint: For a given $\sigma \in S_N$ define the operator $T_\sigma : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ by the formula $T_\sigma\psi(x_1, \dots, x_N) = \psi(x_{\sigma(1)}, \dots, x_{\sigma(N)})$ and observe that it is a unitary operator on $L^2(\mathbb{R}^{3N})$. Then show that the set $L^2_a(\mathbb{R}^{3N})$ is closed in $L^2(\mathbb{R}^{3N})$.)
2. The operator $P_{a,N}\psi = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^\sigma T_\sigma\psi$ is the orthogonal projection onto $L^2_a(\mathbb{R}^{3N})$.
3. $H_{N,Z}(L^2_a(\mathbb{R}^{3N}) \cap H^2(\mathbb{R}^{3N})) \subset L^2_a(\mathbb{R}^{3N})$.
4. The operator $H_{N,Z} : L^2_a(\mathbb{R}^{3N}) \cap H^2(\mathbb{R}^{3N}) \to L^2_a(\mathbb{R}^{3N})$ is self-adjoint. (Hint: Either rework the proof of Exercise 13 of last semester or use that $H_{N,Z} : H^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ is self-adjoint, shown in last semester, together with the basic criterion of self-adjointness.)
5. $\sigma_{ess}(H_{N,Z}|_{L^2_a}) = [\inf \sigma(H_{N-1,Z}|_{L^2_a}), \infty)$. (Hint: With the help of the projection operator $P_{a,N}$ rework the proof of the HVZ theorem, i.e. Theorem 8.6 of last semester which can be found here.)