Mathematical Methods in Quantum Mechanics II

10th Exercise Sheet

Exercise 30:
Consider the quadratic form

$$q_{H_n^{U,\alpha}}(\phi,\psi) = \sum_{j=1}^{n}\Big[\Big\langle \nabla_{x_j}\phi, \nabla_{x_j}\psi\Big\rangle + \Big\langle \phi, a(f_{x_j})\psi\Big\rangle + \Big\langle a(f_{x_j})\phi, \psi\Big\rangle\Big] +$$

$$\Big\langle \phi, \sum_{1\le i<j\le n}\frac{U}{|x_i-x_j|}\psi\Big\rangle + \Big\langle \phi, N\psi\Big\rangle$$

where $f_x(k) = \frac{\sqrt{\alpha}e^{-ikx}}{\sqrt{2}\pi|k|}$ and $\alpha, U > 0$ that is associated with the Hamiltonian operator $H_n^{U,\alpha}$ defined in class. Following the argument that was presented for the case $n = 1$ derive an expression for the operator

$$P_n^{U,\alpha}(\psi) = \inf_{\eta\in\mathcal{F}_{s,c}(L^2(\mathbb{R}^3)),\|\eta\|=1} q_{H_n^{U,\alpha}}(\psi\otimes\eta, \psi\otimes\eta)$$

where $\psi \in C_c^\infty(\mathbb{R}^{3n})$. More precisely, show that

$$P_n^{U,\alpha}(\psi) = \sum_{j=1}^{n}\|\nabla_j\psi\|_2^2 + \Big\langle \psi, \sum_{1\le i<j\le n}\frac{U}{|x_i-x_j|}\psi\Big\rangle - D\alpha\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}\,dxdy \qquad (1)$$

where

$$\rho(x) = \sum_{j=1}^{n}\int_{\mathbb{R}^{3(n-1)}}\Big|\psi(x_1,\dots,x_{j-1},x,x_{j+1}\dots,x_n)\Big|^2\,dx_1\dots dx_{j-1}\hat{dx_j}dx_j\dots dx_n$$

and D is a constant.

Exercise 31:
Define

$$\mathcal{E}_n^{U,\alpha} = \inf_{\|\psi\|_2=1} P_n^{U,\alpha}(\psi)$$

where $P_n^{U,\alpha}$ is as in (1). Show that for all $k \in \{1,\dots,n-1\}$ we have the relation

$$\mathcal{E}_n^{U,\alpha} \le \mathcal{E}_k^{U,\alpha} + \mathcal{E}_{n-k}^{U,\alpha}.$$