Mathematical Methods in Quantum Mechanics II

10th Exercise Sheet

Exercise 30:
Consider the quadratic form

\[ q_{H_n^{U,\alpha}}(\phi, \psi) = \sum_{j=1}^{n} \left[ \langle \nabla x_j \phi, \nabla x_j \psi \rangle + \langle \phi, a(f_{x_j})\psi \rangle + \langle a(f_{x_j})\phi, \psi \rangle \right] + \langle \phi, \sum_{1 \leq i < j \leq n} \frac{U}{|x_i - x_j|}\psi \rangle + \langle \phi, N\psi \rangle \]

where \( f_{x}(k) = \frac{\sqrt{\alpha e^{-ikx}}}{\sqrt{2\pi}|k|} \) and \( \alpha, U > 0 \) that is associated with the Hamiltonian operator \( H_n^{U,\alpha} \) defined in class. Following the argument that was presented for the case \( n = 1 \) derive an expression for the operator

\[ P_{n}^{U,\alpha}(\psi) = \inf_{\eta \in F_{s,c}(L^2(R^3)), \|\eta\| = 1} q_{H_n^{U,\alpha}}(\psi \otimes \eta, \psi \otimes \eta) \]

where \( \psi \in C^\infty_c(R^{3n}) \). More precisely, show that

\[ P_{n}^{U,\alpha}(\psi) = \sum_{j=1}^{n} \|\nabla_j \psi\|_2^2 + \left\langle \psi, \sum_{1 \leq i < j \leq n} \frac{U}{|x_i - x_j|}\psi \right\rangle - D\alpha \int_{R^3} \int_{R^3} \frac{\rho(x)\rho(y)}{|x - y|} \, dx \, dy \quad (1) \]

where

\[ \rho(x) = \sum_{j=1}^{n} \int_{R^{3(n-1)}} \left| \psi(x_1, \ldots, x_{j-1}, x, x_{j+1} \ldots, x_n) \right|^2 \, dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n \]

and \( D \) is a constant.

Exercise 31:
Define

\[ \mathcal{E}_{n}^{U,\alpha} = \inf_{\|\psi\|_2=1} P_{n}^{U,\alpha}(\psi) \]

where \( P_{n}^{U,\alpha} \) is as in (1). Show that for all \( k \in \{1, \ldots, n - 1\} \) we have the relation

\[ \mathcal{E}_{n}^{U,\alpha} \leq \mathcal{E}_{k}^{U,\alpha} + \mathcal{E}_{n-k}^{U,\alpha} \]

http://www.math.kit.edu/iana1/edu/mathmethquantmech2020s/