Institut für Analysis
Dr. Ioannis Anapolitanos
Dr. Nikolaos Pattakos

SS2020
29/04/2020

Mathematical Methods in Quantum Mechanics II

2nd Exercise Sheet

Exercise 2:

Suppose that A is a bounded operator on a Hilbert space $\mathcal{H}$. Show that there exists a partial isometry V on $\mathcal{H}$ such that

$$A = V|A| \text{ and } |A| = V^*A.$$

Such decomposition of A is called the polar decomposition of A. (Hint: Observe that $\mathrm{Ker}(|A|) = \mathrm{Ker}(A)$ and therefore the map $V : \mathrm{Ran}(|A|) \to \mathrm{Ran}(A)$, $V(|A|x) = Ax$ for $x \in \mathcal{H}$ is well-defined.)

Exercise 3:

Suppose that A is a bounded operator on a Hilbert space $\mathcal{H}$ and U partial isometry on $\mathcal{H}$. Show that

$$\mathrm{Tr}(U^*|A|U) \leq \mathrm{Tr}(|A|).$$

Conclude that if V is another partial isometry of $\mathcal{H}$ then $\mathrm{Tr}(U^*V|A|V^*U) \leq \mathrm{Tr}(|A|)$. (Hint: Choose a suitable orthonormal basis.)

If in addition, $A \in \mathcal{L}^1(\mathcal{H})$ and B is a bounded operator on $\mathcal{H}$ then

$$\|AB\|_{\mathcal{L}^1(\mathcal{H})} \leq \|A\|_{\mathcal{L}^1(\mathcal{H})} \cdot \|B\|.$$

(Hint: Use the polar decomposition of A and AB and follow the ideas of the proof of Theorem 12.4.a.)

Exercise 4:

Consider $A \in \mathcal{L}^1(\mathcal{H})$ and $\{\phi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}}$ ONB in the Hilbert space $\mathcal{H}$. Show that

$$\sum_{n=1}^{\infty}\langle \phi_n, A\phi_n\rangle = \sum_{n=1}^{\infty}\langle \psi_n, A\psi_n\rangle.$$

Exercise 5:

Show that the map

$$T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}^1(\mathcal{H})^*,\ TA = \mathrm{Tr}(A\cdot),\ A \in \mathcal{L}(\mathcal{H})$$

is an isometric isomorphism. (Hint: Consider $g \in \mathcal{L}^1(\mathcal{H})^*$ and show with the use of Riesz representation theorem that there is a unique operator $B \in \mathcal{L}(\mathcal{H})$ with the property

$$\langle \psi, B\phi\rangle = g\Big(\langle \psi, \cdot\rangle\phi\Big)$$

for all $\phi, \psi \in \mathcal{H}$.

Then prove that $A \mapsto \mathrm{Tr}(BA)$ is a bounded linear functional on $\mathcal{L}^1(\mathcal{H})$ which agrees with g and that $\|g\|_{\mathcal{L}^1(\mathcal{H})^*} = \|B\|$.)