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SS2020
08/05/2020

Mathematical Methods in Quantum Mechanics II

3rd Exercise Sheet

Exercise 6:
Consider two Hilbert spaces $\mathcal{H}^1, \mathcal{H}^2$ and an ONB $\{\phi_n\}_{\mathbb{N}} \subset \mathcal{H}^1$. For $M \in \mathbb{N}$ define the orthogonal projection operator

$$P_M = \sum_{m=1}^{M} |\phi_m\rangle\langle\phi_m|.$$

Show that if $\rho \in \mathcal{L}^1(\mathcal{H}^1)$ then

$$\lim_{M\to\infty} \|\rho - P_M\rho\|_{\mathcal{L}^1(\mathcal{H}^1)} = 0.$$

In addition, if $R \in \mathcal{L}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$ prove that

$$\lim_{M\to\infty} \|R - (P_M \otimes \mathrm{Id})R\|_{\mathcal{L}^1(\mathcal{H}^1\otimes\mathcal{H}^2)} = 0.$$

Exercise 7:
We consider the Hartree equation

$$\begin{cases} i\psi_t = -\Delta\psi + (V * |\psi|^2)\psi & , \ (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ \psi(0,x) = \psi_0(x) & , \ x \in \mathbb{R}^d \end{cases} \tag{1}$$

with initial data $\psi_0 \in L^2(\mathbb{R}^d)$, where V is a given function in $L^\infty(\mathbb{R}^d)$ (the convolution above is in the space variable x). Using Duhamel's formula we may rewrite (1) in the integral formulation of the equation

$$\psi(t,x) = e^{it\Delta}\psi_0(x) - i\int_0^t e^{i(t-\tau)\Delta}[(V * |\psi|^2)\psi](\tau,x)\, d\tau. \tag{2}$$

We say that the Hartree equation (1) is locally-wellposed in $L^2(\mathbb{R}^d)$ if for any $\psi_0 \in L^2(\mathbb{R}^d)$ there exists a time $T > 0$ and a unique function $\psi \in C([-T,T], L^2(\mathbb{R}^d))$ satisfying (2) and furthermore the map $\psi_0 \mapsto \psi$ is continuous from $L^2(\mathbb{R}^d)$ to $C([-T,T], L^2(\mathbb{R}^d))$. If we can take T arbitrarily large we say that the wellposedness is global.

Show the following:

1. The Hartree equation is locally wellposed in $L^2(\mathbb{R}^d)$.

   Hint: Consider a $\psi_0 \in L^2(\mathbb{R}^d)$ and the complete metric space

   $$M(R,T) = \Big\{u \in C([-T,T], L^2(\mathbb{R}^d)) \Big| \|u\|_M := \sup_{-T\le t\le T} \|u(t,\cdot)\|_2 \le R\Big\},$$

   where $R, T \geq 0$. Show that for suitably chosen R, T the operator

   $$\mathcal{T}u = e^{it\Delta}\psi_0(x) - i\int_0^t e^{i(t-\tau)\Delta}[(V * |u|^2)u](\tau,x)\, d\tau$$

   is a contraction in $M(R,T)$.

2. The solution ψ conserves the $L^2(\mathbb{R}^d)$ norm, i.e. $\|\psi(t,\cdot)\|_2 = \|\psi_0\|_2$, for all $t \in [-T,T]$. Hint: Use that $e^{-i\Delta t}$ is unitary together with Duhamel's formula to justify the differentiability of $\|\psi(t,\cdot)\|_2^2$ in time and show that its derivative is 0.
3. The Hartree equation is globally wellposed in $L^2(\mathbb{R}^d)$.

   Hint: Use the conservation of the L^2 norm.