Mathematical Methods in Quantum Mechanics II

3rd Exercise Sheet

Exercise 6:
Consider two Hilbert spaces $\mathcal{H}^1, \mathcal{H}^2$ and an ONB $\{\phi_n\}_N \subset \mathcal{H}^1$. For $M \in \mathbb{N}$ define the orthogonal projection operator

$$P_M = \sum_{m=1}^{M} |\phi_m\rangle \langle \phi_m|.$$ 

Show that if $\rho \in \mathcal{L}^1(\mathcal{H}^1)$ then

$$\lim_{M \to \infty} \|\rho - P_M \rho\|_{\mathcal{L}^1(\mathcal{H}^1)} = 0.$$ 

In addition, if $R \in \mathcal{L}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$ prove that

$$\lim_{M \to \infty} \|R - (P_M \otimes \text{Id})R\|_{\mathcal{L}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)} = 0.$$ 

Exercise 7:
We consider the Hartree equation

$$\begin{cases}
  i\psi_t = -\Delta \psi + (V \ast |\psi|^2)\psi, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
  \psi(0, x) = \psi_0(x), & x \in \mathbb{R}^d
\end{cases}$$

(1)

with initial data $\psi_0 \in L^2(\mathbb{R}^d)$, where $V$ is a given function in $L^\infty(\mathbb{R}^d)$ (the convolution above is in the space variable $x$). Using Duhamel’s formula we may rewrite (1) in the integral formulation of the equation

$$\psi(t, x) = e^{it\Delta} \psi_0(x) - i \int_0^t e^{i(t-\tau)\Delta} [V \ast |\psi|^2]u(\tau, x) \, d\tau.$$ 

(2)

We say that the Hartree equation (1) is locally-wellposed in $L^2(\mathbb{R}^d)$ if for any $\psi_0 \in L^2(\mathbb{R}^d)$ there exists a time $T > 0$ and a unique function $\psi \in C([-T, T], L^2(\mathbb{R}^d))$ satisfying (2) and furthermore the map $\psi_0 \mapsto \psi$ is continuous from $L^2(\mathbb{R}^d)$ to $C([-T, T], L^2(\mathbb{R}^d))$. If we can take $T$ arbitrarily large we say that the wellposedness is global.

Show the following:

1. The Hartree equation is locally well posed in $L^2(\mathbb{R}^d)$.

   Hint: Consider a $\psi_0 \in L^2(\mathbb{R}^d)$ and the complete metric space

   $$M(R, T) = \left\{ u \in C([-T, T], L^2(\mathbb{R}^d)) \mid \|u\|_M := \sup_{-T \leq \tau \leq T} \|u(t, \cdot)\|_2 \leq R \right\},$$

where $R, T \geq 0$. Show that for suitably chosen $R, T$ the operator

   $$\mathcal{T}u = e^{it\Delta} \psi_0(x) - i \int_0^t e^{i(t-\tau)\Delta} [V \ast |u|^2]u(\tau, x) \, d\tau$$

is a contraction in $M(R, T)$. 

2. The solution $\psi$ conserves the $L^2(\mathbb{R}^d)$ norm, i.e. $\|\psi(t, \cdot)\|_2 = \|\psi_0\|_2$, for all $t \in [-T, T]$. 
   Hint: Use that $e^{-it\Delta}$ is unitary together with Duhamel’s formula to justify the differentiability of $\|\psi(t, \cdot)\|_2^2$ in time and show that its derivative is 0.

3. The Hartree equation is globally wellposed in $L^2(\mathbb{R}^d)$.
   Hint: Use the conservation of the $L^2$ norm.