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Mathematical Methods in Quantum Mechanics II

4th Exercise Sheet

Exercise 8:

Let $\mathcal{H}$ be a separable Hilbert space and define

$$\mathcal{L}^2(\mathcal{H}) = \Big\{ K \in \mathcal{L}(\mathcal{H}) \;\Big|\; \mathrm{Tr}(K^*K) < \infty \Big\}.$$

Show the following:

1. The map $(K_1, K_2) \mapsto \langle K_1, K_2 \rangle_{\mathcal{L}^2(\mathcal{H})} := \mathrm{Tr}(K_1^* K_2)$ for $K_1, K_2 \in \mathcal{L}^2(\mathcal{H})$ defines an inner product in $\mathcal{L}^2(\mathcal{H})$ and therefore, the expression
$$\|K\|_{\mathcal{L}^2(\mathcal{H})} = \sqrt{\mathrm{Tr}(K^*K)}$$
defines a norm in $\mathcal{L}^2(\mathcal{H})$.
2. If $K \in \mathcal{L}^1(\mathcal{H})$ then $K \in \mathcal{L}^2(\mathcal{H})$ and actually
$$\|K\|_{\mathcal{L}^2(\mathcal{H})} \le \|K\|_{\mathcal{L}^1(\mathcal{H})}.$$

Exercise 9:

Consider two separable Hilbert spaces $\mathcal{H}^1, \mathcal{H}^2$ and $R \in \mathcal{L}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$. By Theorem 12.11 we know that there is a unique $\rho \in \mathcal{L}^1(\mathcal{H}^1)$ (called the partial trace of R) such that

$$\mathrm{Tr}((b \otimes \mathrm{Id})R) = \mathrm{Tr}(b\rho)$$

for all $b \in \mathcal{L}(\mathcal{H}^1)$. Show that

$$\mathrm{Tr}(|\rho|) \le \mathrm{Tr}(|R|).$$

Exercise 10:

Let $\psi_N \in L^2_s(\mathbb{R}^{3N})$, $\psi \in L^2(\mathbb{R}^3)$ with $\|\psi_N\|_2 = \|\psi\|_2 = 1$ and denote by $\gamma_N^{(p)}$, $p \in \{1, \ldots, N\}$, the partial trace of the projection $|\psi_N\rangle\langle\psi_N|$ over the $p+1, \ldots, N$ particle coordinates. Define the projection operators

$$p_j^{\psi} \psi_N(x_1, \ldots, x_N) = \psi(x_j) \int_{\mathbb{R}^3} \overline{\psi(y_j)}\; \psi_N(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_N)\, dy_j,$$

$j \in \{1, \ldots, N\}$ and $q_j^{\psi} = 1 - p_j^{\psi}$. Show that

$$\mathrm{Tr}\big|\gamma_N^{(p)} - |\psi^{\otimes p}\rangle\langle\psi^{\otimes p}|\big| \le \sqrt{8p \cdot c_N(\psi_N, \psi)}$$

where

$$c_N(\psi_N, \psi) = \frac{1}{N} \Big\langle \psi_N, \sum_{j=1}^{N} q_j^{\psi} \psi_N \Big\rangle.$$

Exercise 11:
The goal of this exercise is to fill in the details of the heuristic Geometric argument presented in the lecture for the derivation of the Hartree equation from the many particle system

$$\begin{cases} i\partial_t \Psi_{N,t} = H_N \Psi_{N,t} \\ \Psi_{N,0} = \psi_0^{\otimes N} \in L^2_s(\mathbb{R}^{3N}) \end{cases}$$

where $H_N = -\sum_{j=1}^N \Delta_{x_j} + \alpha \sum_{1\leq i<j\leq N} V(x_i - x_j)$ with $\alpha \in \mathbb{R}$ and the potential V is even.

Following the lecture, in $L^2(\mathbb{R}^{3N})$ we consider the manifold

$$M = \{\phi^{\otimes N} \in L^2(\mathbb{R}^{3N}) \;\Big|\; \phi \in L^2(\mathbb{R}^3),\ \|\phi\|_2 = 1\}.$$

For a point $\psi_0^{\otimes N} \in M$ we consider the tangent space in M

$$T_{\psi_0^{\otimes N}} = \Big\{ \frac{d}{dt}(\gamma(t)^{\otimes N})|_{t=0} \;\Big|\; \gamma \in C^1((-1,1), L^2(\mathbb{R}^3)),\ \gamma(0) = \psi_0,\ \|\gamma(t)\|_2 = 1\ \forall t \in (-1,1) \Big\}.$$

Follow the steps:

1. If $\dot{\psi} \in L^2(\mathbb{R}^3)$ with $\dot{\psi} \perp \psi_0$ then the expression
$$\sum_{j=1}^N \psi_0^{\otimes(j-1)} \otimes \dot{\psi} \otimes \psi_0^{\otimes(N-j)} \tag{1}$$
is a tangent vector in M at the tangent space $T_{\psi_0^{\otimes N}}$. Conversely, every tangent vector can be written as in (1) for some $\dot{\psi} \in L^2(\mathbb{R}^3)$ with $\dot{\psi} \perp \psi_0$.
2. Let $V_{12}(x_1, x_2) = V(x_1 - x_2)$ and consider $\phi, \psi_0 \in L^2(\mathbb{R}^3)$. Show that
$$\langle V_{12}\psi_0^{\otimes 2}, \phi \otimes \psi_0 \rangle_{L^2(\mathbb{R}^6)} = \langle (V * |\psi_0|^2)\psi_0, \phi \rangle_{L^2(\mathbb{R}^3)}.$$
3. For $\psi_0^{\otimes N}$ and $\phi \perp \psi_0$ calculate the inner product $\frac{1}{N}\langle -iH_N\psi_0^{\otimes N}, T_{\psi^{\otimes N}}\phi\rangle$. For which value of α is it independent of N?
4. For this value of α and for $\psi_0^{\otimes N} \in M$ calculate the orthogonal projection of $-iH_N\psi_0^{\otimes N}$ onto $T_{\psi_0^{\otimes N}}$. Doing the above and the procedure presented in class for all initial data of the form $\psi_0^{\otimes N}$ we obtain a vector field on the manifold M. Let $\psi_t^{\otimes N}$ be the dynamics induced by this vector field, if $\psi_t\Big|_{t=0} = \psi_0$. Show that ψ_t solves, up to a time dependent factor, the Hartree equation.