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Mathematical Methods in Quantum Mechanics II

5th Exercise Sheet

Exercise 12:
Let $\mathcal{H}$ be a separable Hilbert space and consider an operator $K \in \mathcal{L}(\mathcal{H})$. Show that

$$\|K\|_{\mathcal{L}(\mathcal{H})} \leq \|K\|_{\mathcal{L}^2(\mathcal{H})} \leq \|K\|_{\mathcal{L}^1(\mathcal{H})}.$$

(Hint: By HW4 Exercise 8 part 2 it suffices to show only the left inequality.)

Exercise 13:
Consider $N \in \mathbb{N}$, $\psi \in L^2(\mathbb{R}^3)$ with $\|\psi\|_2 = 1$ and $V \in L^\infty(\mathbb{R}^3)$. Define $V_{12}(x_1, x_2) = V(x_1 - x_2)$, for $x_1, x_2 \in \mathbb{R}^3$ and the operators

$$Tf = V_{12}f, \; Sf = (V * |\psi|^2)f$$

for $f \in L^2(\mathbb{R}^{3N})$. Show that

$$\|T\|_{L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})}, \; \|S\|_{L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})} \leq \|V\|_\infty.$$

Exercise 14:
For a measurable function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ we formally define the integral operator

$$L_k f(x) = \int_{\mathbb{R}^d} k(x, y) f(y) \; dy.$$

Show the following:

1. If $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ then the integral operator L_k is Hilbert-Schmidt in $L^2(\mathbb{R}^d)$ and we have
$$\|L_k\|_{\mathcal{L}^2(L^2(\mathbb{R}^d))} = \|k\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}.$$
2. If T is a Hilbert-Schimdt operator on $L^2(\mathbb{R}^d)$ then there exists a function $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that
$$T = L_k.$$
The function k is called the kernel of the operator T. (Hint: Try to identify the function k using sums of ONB. More precisely, you need to use that if $\{e_n\}_n$ is an ONB of $L^2(\mathbb{R}^d)$ then the sequence $\{e_m \otimes \overline{e_n}\}_{m,n}$ is an ONB of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Here the notion of unconditional convergence of a sequence $\{x_n\}_n$ in a Hilbert space comes into play. See page 2 for more details.)
3. The map $k \mapsto L_k$ is an isometric isomorphism of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ onto $\mathcal{L}^2(L^2(\mathbb{R}^d))$.
4. If $T \in \mathcal{L}^1(L^2(\mathbb{R}^d))$ with continuous kernel k (actually the continuity assumption is not necessary) then
$$\mathrm{Tr}(T) = \int_{\mathbb{R}^d} k(x, x) \; dx.$$
(In this question show the above equality non-rigorously.)

5. Let $R \in \mathcal{L}^1(L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^m))$ and $\rho \in \mathcal{L}^1(L^2(\mathbb{R}^d))$ be its partial trace. If $R((x,y),(v,w))$, $x, v \in \mathbb{R}^d$, $y, w \in \mathbb{R}^m$, is the kernel of R show that

$$\rho(x,v) = \int_{\mathbb{R}^m} R((x,y),(v,y))\, dy$$

is the kernel of ρ.

Definition of **unconditional convergence** and **absolute convergence**: Consider a sequence $\{x_n\}_n$ in a Banach space $(X, \|\cdot\|)$. We say that the series $\sum_n x_n$

1. is unconditionally convergent if the series $\sum_n x_{\sigma(n)}$ converges in X for every permutation σ of the natural numbers $\mathbb{N}$.
2. is absolutely convergent if $\sum_n \|x_n\| < \infty$.

If X is finite dimensional then these two notions coincide. But in an infinite dimensional Banach space X the two notions are not the same. It is true that if a sequence is absolutely convergent then it is also unconditionally convergent but the converse is not true in general. In addition, if a series is unconditionally convergent it can be proved that the value $\sum_n x_{\sigma(n)}$ is independent of the permutation σ.

In Exercise 14, Question 2 you will need that in a Hilbert space H with an ONB $\{e_n\}_n$ and a sequence of scalars $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ the following are equivalent:

1. $\sum_n c_n e_n$ converges in H.
2. $\sum_n c_n e_n$ converges unconditionally in H.
3. $\sum_n |c_n|^2 < \infty$.