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Mathematical Methods in Quantum Mechanics II

6th Exercise Sheet

Exercise 15:
Suppose that h is a complex separable Hilbert space (this assumption holds in all of the following exercises too). Show that the number operator $N : D(N) \to \mathcal{F}(h)$ is self-adjoint, where

$$D(N) = \Big\{ \psi = \{\psi_n\}_{n\in\mathbb{N}_0} \in \bigoplus_{n=0}^{\infty} h^{\otimes n} \ \Big|\ \sum_{n=0}^{\infty} \|n\psi_n\|_{h_n}^2 < \infty \Big\}.$$

Then prove that $N : P_{s,a}D(N) \to \mathcal{F}_{s,a}(h)$ is also self-adjoint.

Exercise 16:
Show that

$$\Big\langle \phi, a(f)\psi \Big\rangle = \Big\langle a^*(f)\phi, \psi \Big\rangle$$

for all $\phi, \psi \in D(N^{\frac{1}{2}})$ and $f \in h$ where $\Big\langle \cdot,\cdot \Big\rangle$ is the inner product in the Fock space $\mathcal{F}(h)$.

Exercise 17:
Show that for all $f \in h$ we have

$$a_a^*(f)a_a^*(f) = 0$$

on $D(N)$.

Exercise 18:
Show that for all $f \in L^2(\mathbb{R}^3)$ and $\psi \in L_a^2(\mathbb{R}^{3N})$

$$\Big(a_a(f)\psi\Big)(x_1, \ldots, x_{N-1}) = (-1)^{N-1}\ N^{\frac{1}{2}} \int_{\mathbb{R}^3} \overline{f(x_N)}\ \psi(x_1, \ldots, x_N)\ dx_N$$

and

$$\Big(a_a^*(f)\psi\Big)(x_1 \ldots, x_{N+1}) = (N+1)^{-\frac{1}{2}} \sum_{j=1}^{N+1} (-1)^{j+1}\ f(x_j)\ \psi(x_1, \ldots, \hat{x}_j \ldots, x_{N+1})$$

where $x_1, \ldots, x_{N+1} \in \mathbb{R}^3$.

Exercise 19:
Prove the canonical anti-commutation relations on $\mathcal{F}_a(L^2(\mathbb{R}^3))$, that is show

$$\{a^*(f), a^*(g)\} = \{a(f), a(g)\} = 0 \text{ and } \{a(f), a^*(g)\} = \Big\langle f, g \Big\rangle_{L^2(\mathbb{R}^3)} \mathrm{Id}$$

for all $f, g \in L^2(\mathbb{R}^3)$.