Mathematical Methods in Quantum Mechanics II

7th Exercise Sheet

Exercise 20:
Prove that the operators \( a(f), a^*(f) \), where \( f \in L^2(\mathbb{R}^3) \), are unbounded on \( \mathcal{F}_s(L^2(\mathbb{R}^3)) \) but bounded on \( \mathcal{F}_a(L^2(\mathbb{R}^3)) \).

Exercise 21:
Show that a semi-bounded quadratic form \( q : Q(q) \times Q(q) \to \mathbb{C} \) is closed if and only if for every sequence \( \{\phi_n\}_n \subset Q(q) \) and \( \phi \in \mathcal{H} \) with \( \phi_n \to \phi \) in \( \mathcal{H} \) and \( q(\phi_n - \phi_m, \phi_n - \phi_m) \to 0 \) as \( n, m \to \infty \) we have that \( \phi \in Q(q) \) and \( q(\phi_n - \phi, \phi_n - \phi) \to 0 \) as \( n \to \infty \).

Using the above criterion show that the quadratic form \( q : C_c(\mathbb{R}) \times C_c(\mathbb{R}) \to \mathbb{C} \) in \( L^2(\mathbb{R}) \) given by \( q(f, g) = f(0)g(0) \) is not closed. Moreover, prove that \( q \) does not have any closed extensions.

Exercise 22:
Show that the quadratic form \( q(\phi, \psi) + \frac{5}{4}\langle \phi, \psi \rangle \) where \( q : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \to \mathbb{C} \) is defined as

\[
q(\phi, \psi) = \int_{\mathbb{R}^3} \left[ \nabla \phi(x) \cdot \nabla \psi(x) - \frac{\phi(x) \psi(x)}{|x|} \right] \, dx
\]

is associated with the operator \(-\Delta - \frac{1}{|x|} + \frac{5}{4}\).