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Mathematical Methods in Quantum Mechanics II

8th Exercise Sheet

Exercise 23:
In the Hilbert space $L^2(\mathbb{R})$ we consider the quadratic form $q_H : H^{1,1}(\mathbb{R}) \times H^{1,1}(\mathbb{R}) \to \mathbb{C}$ given by the formula

$$q_H(\phi, \psi) = \frac{1}{2}\Big[\Big\langle \frac{d}{dx}\phi, \frac{d}{dx}\psi \Big\rangle_{L^2(\mathbb{R})} + \Big\langle x\phi, x\psi \Big\rangle_{L^2(\mathbb{R})}\Big].$$

Show that q_H is closed.

Exercise 24:
Let $q : Q(q) \times Q(q) \to \mathbb{C}$ be a quadratic form in a Hilbert space $\mathcal{H}$. Show that if q is closable and semi-bounded then its closure $\bar{q} : Q(\bar{q}) \times Q(\bar{q}) \to \mathbb{C}$ is semi-bounded with the same lower bound as q.

Exercise 25:
Consider the quadratic form $q : C_c(\mathbb{R}) \times C_c(\mathbb{R}) \to \mathbb{C}$ in $L^2(\mathbb{R})$ given by $q(f, g) = \overline{f(0)}g(0)$. Show that the completion of the normed space $(C_c(\mathbb{R}), \|\cdot\|_q)$, where

$$\|f\|_q = \Big(|f(0)|^2 + \|f\|_2^2\Big)^{\frac{1}{2}} \text{ for } f \in C_c(\mathbb{R}),$$

is not a subset of $L^2(\mathbb{R})$.

Exercise 26:
Let

$$\psi_0(x) = \frac{e^{-\frac{x^2}{2}}}{\pi^{\frac{1}{4}}}$$

and define $\psi_n(x) = (a^*)^{(n)}\psi_0(x)$, where $n \in \mathbb{N}$ (for an operator A we denote by $A^{(n)}$ the composition with itself n times). Show the following:

1. $\|\psi_n\|_2 = \sqrt{n!}$ for all $n \in \mathbb{N}$.
2. If ψ is an eigenfunction of the operator $N = a^*a$ and if $a^{(n)}\psi = C\psi_0$ for some $C \in \mathbb{C}$ then $\psi = D\psi_n$ for some $D \in \mathbb{C}$.

(Hint: For the first question use induction in n and that $[a, a^*] = \mathrm{Id}$. For the second question observe that if $(\frac{d}{dx} + x)^{(n)}\psi = ce^{-\frac{x^2}{2}}$ then $(\frac{d}{dx})^{(n)}(e^{\frac{x^2}{2}}\psi) = c$.)