Mathematical Methods in Quantum Mechanics II

8th Exercise Sheet

Exercise 23:
In the Hilbert space $L^2(\mathbb{R})$ we consider the quadratic form $q_H : H^{1,1}(\mathbb{R}) \times H^{1,1}(\mathbb{R}) \to \mathbb{C}$ given by the formula

$$q_H(\phi, \psi) = \frac{1}{2} \left( \left\langle \frac{d}{dx} \phi, \frac{d}{dx} \psi \right\rangle_{L^2(\mathbb{R})} + \left\langle x\phi, x\psi \right\rangle_{L^2(\mathbb{R})} \right).$$

Show that $q_H$ is closed.

Exercise 24:
Let $q : Q(q) \times Q(q) \to \mathbb{C}$ be a quadratic form in a Hilbert space $H$. Show that if $q$ is closable and semi-bounded then its closure $\bar{q} : Q(\bar{q}) \times Q(\bar{q}) \to \mathbb{C}$ is semi-bounded with the same lower bound as $q$.

Exercise 25:
Consider the quadratic form $q : C_c(\mathbb{R}) \times C_c(\mathbb{R}) \to \mathbb{C}$ in $L^2(\mathbb{R})$ given by $q(f, g) = f(0)g(0)$. Show that the completion of the normed space $(C_c(\mathbb{R}), \| \cdot \|_q)$, where

$$\|f\|_q = \left( |f(0)|^2 + \|f\|_2^2 \right)^{1/2} \text{ for } f \in C_c(\mathbb{R}),$$

is not a subset of $L^2(\mathbb{R})$.

Exercise 26:
Let $\psi_0(x) = e^{-x^2/2}$ and define $\psi_n(x) = (a^*)^n \psi_0(x)$, where $n \in \mathbb{N}$ (for an operator $A$ we denote by $A^{(n)}$ the composition with itself $n$ times). Show the following:

1. $\|\psi_n\|_2 = \sqrt{n!}$ for all $n \in \mathbb{N}$.
2. If $\psi$ is an eigenfunction of the operator $N = a^*a$ and if $a^{(n)} \psi = C\psi_0$ for some $C \in \mathbb{C}$ then $\psi = D\psi_n$ for some $D \in \mathbb{C}$.

(Hint: For the first question use induction in $n$ and that $[a, a^*] = \text{Id}$. For the second question observe that if $(\frac{d}{dx} + x)^{(n)} \psi = c e^{-x^2/2}$ then $(\frac{d}{dx})^{(n)}(e^{x^2/2} \psi) = c$.)