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SS2020
17/06/2020

Mathematical Methods in Quantum Mechanics II

9th Exercise Sheet

Exercise 27:
Compute

$$\mathcal{F}^{-1}a^*(k)\mathcal{F} = c\int e^{ikx}a^*(x)dx \text{ and } \mathcal{F}^{-1}\int a^*(k)a(k)dk\mathcal{F} = d\int a^*(x)a(x)dx$$

where $\mathcal{F}$ is the Fourier transform and a^*, a the creation and annihilation operators respectively on the Fock space $\mathcal{F}_s(L^2(\mathbb{R}^3))$ and c, d are constants.

Exercise 28:
Let c be constant and define $g_{x,1}(k) = c\,\frac{ke^{-ikx}}{|k|^3}\,\chi_{\{|k|\geq 1\}}(k)$ and $f_{x,1,\infty}(k) = i\text{div}g_{x,1}(k)$ where $k, x \in \mathbb{R}^3$. Show that

$$\langle \phi, a(f_{x,1,\infty})\phi\rangle = \sum_{j=1}^{3}\left[\langle -i\partial_{x_j}\phi, a(g_{x,1}^{(j)})\phi\rangle - \langle a^*(g_{x,1}^{(j)})\phi, -i\partial_{x_j}\phi\rangle\right]$$

for any $\phi \in C_c^\infty(\mathbb{R}^3)\otimes\mathcal{F}_{s,c}(L^2(\mathbb{R}^3))$.

Exercise 29:
Define the set $Q_H = C_c^\infty(\mathbb{R}^{3n})\otimes\mathcal{F}_{s,c}(L^2(\mathbb{R}^3))$ and the quadratic form $q_H : Q_H \times Q_H \to \mathbb{C}$ given by

$$q_H(\phi,\psi) = \sum_{j=1}^{n}\left[\left\langle \nabla_{x_j}\phi, \nabla_{x_j}\psi\right\rangle + \left\langle \phi, a(f_{x_j})\psi\right\rangle + \left\langle a(f_{x_j})\phi, \psi\right\rangle\right] +$$

$$\left\langle \phi, \sum_{1\leq i<j\leq n}\frac{U}{|x_i - x_j|}\psi\right\rangle + \left\langle \phi, N\psi\right\rangle$$

where $f_x(k) = \frac{\sqrt{\alpha}e^{-ikx}}{\sqrt{2}\pi|k|}$ and $\alpha, U > 0$. Show that q_H is a closable and semi-bounded quadratic form.