Exercise 30. By definition $P_n^{U,\alpha}(\psi)$ is the infimum over $\eta \in \mathcal{F}_{s,c}(L^2(\mathbb{R}^3))$, $\|\eta\| = 1$ of the expression

$$(1) \qquad \sum_{j=1}^n \|\nabla_j \psi\|_2^2 + \Big\langle \psi, \sum_{1\le i<j\le n} \frac{U}{|x_i - x_j|}\psi \Big\rangle + \\ \Big\langle \psi\otimes\eta, \Big(\int a^*(k)a(k)dk - c\sqrt{\alpha}\sum_{j=1}^n \int \frac{e^{-ikx_j}a^*(k) + e^{ikx_j}a(k)}{|k|} dk\Big)(\psi\otimes\eta)\Big\rangle.$$

From the definition of the function ρ we have

$$\rho(x) = \sum_{j=1}^n \int_{\mathbb{R}^{3(n-1)}} \Big|\psi(x_1,\dots,x_{j-1},x,x_{j+1}\dots,x_n)\Big|^2 \, dx_1\dots dx_{j-1}d\hat{x}_j dx_j \dots dx_n$$

and thus, its Fourier transform is given by

$$\hat{\rho}(k) = \sum_{j=1}^n \int_{\mathbb{R}^{3n}} e^{-ikx}\Big|\psi(x_1,\dots,x_{j-1},x,x_{j+1}\dots,x_n)\Big|^2 dx_1\dots dx_{j-1}dx dx_j \dots dx_n.$$

This allows us to rewrite the last term of (1) as

$$\Big\langle \eta, \Big(\int a^*(k)a(k)dk - c\sqrt{\alpha}\int \frac{\hat{\rho}(k)a^*(k) + \bar{\hat{\rho}}(k)a(k)}{|k|}dk\Big)\eta\Big\rangle$$

and then proceed in exactly the same way as it was presented in class. Under the assumption that there is an $\eta \in \mathcal{F}_{s,c}(L^2(\mathbb{R}^3))$, $\|\eta\| = 1$ with the property

$$(2) \qquad \Big\langle \eta, \int dk\Big(a^*(k) - c\sqrt{\alpha}\frac{\bar{\hat{\rho}}(k)}{|k|}\Big)\Big(a(k) - c\sqrt{\alpha}\frac{\hat{\rho}(k)}{|k|}\Big)\eta\Big\rangle = 0$$

we obtain the desired equality for the operator $P_n^{U,\alpha}$.

□

Exercise 31. Since U and α are fixed we drop them from the notation. Consider an $\epsilon > 0$. We may find $\phi \in C_c^\infty(\mathbb{R}^{3k})$ and $\psi \in C_c^\infty(\mathbb{R}^{3(n-k)})$ such that $\|\phi\|_2 = \|\psi\|_2 = 1$ and

$$(3) \qquad P_k(\phi) < \mathcal{E}_k + \frac{\epsilon}{2} \text{ and } P_{n-k}(\psi) < \mathcal{E}_{n-k} + \frac{\epsilon}{2}.$$

Let us denote by $\psi_h = \psi(\cdot - h)$ for $h \in \mathbb{R}^{3(n-k)}$ and observe that $\phi\otimes\psi_h \in C_c^\infty(\mathbb{R}^n)$, $\|\phi\otimes\psi_h\|_2 = 1$. We claim

$$(4) \qquad \lim_{|h|\to\infty} P_n(\phi\otimes\psi_h) = P_k(\phi) + P_{n-k}(\psi).$$

Indeed,

$$(5) \qquad P_n(\phi\otimes\psi_h) = \sum_{j=1}^n \|\nabla_j(\phi\otimes\psi_h)\|_2^2 + \Big\langle \phi\otimes\psi_h, \sum_{1\le i<j\le n}\frac{U}{|x_i - x_j|}(\phi\otimes\psi_h)\Big\rangle + \\ -D\alpha \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_{\phi\otimes\psi_h}(x)\rho_{\phi\otimes\psi_h}(y)}{|x-y|}\, dxdy$$

where

$$\rho_{\phi\otimes\psi_h}(x) = \sum_{j=1}^{n} \int_{\mathbb{R}^{3(n-1)}} \left|(\phi \otimes \psi_y)(x_1, \ldots, x_{j-1}, x, x_{j+1} \ldots, x_n)\right|^2 \, dx_1 \ldots dx_{j-1} d\hat{x}_j dx_j \ldots dx_n$$

and D is a constant.

For the first term of the RHS of (5) it is trivial to see that ($\|\phi\|_2 = \|\psi\|_2 = 1$)

$$\sum_{j=1}^{n} \|\nabla_j(\phi \otimes \psi_h)\|^2_{L^2(\mathbb{R}^{3n})} = \sum_{j=1}^{k} \|\nabla_j \phi\|^2_{L^2(\mathbb{R}^{3k})} + \sum_{j=k+1}^{n} \|\nabla_j \psi_h\|^2_{L^2(\mathbb{R}^{3(n-k)})} = \sum_{j=1}^{k} \|\nabla_j \phi\|^2_{L^2(\mathbb{R}^{3k})} + \sum_{j=k+1}^{n} \|\nabla_j \psi\|^2_{L^2(\mathbb{R}^{3(n-k)})}. \tag{6}$$

For the second term of the RHS of (5) we have

$$\left\langle \phi \otimes \psi_h, \sum_{1\le i<j\le n} \frac{U}{|x_i - x_j|}(\phi \otimes \psi_h)\right\rangle = \left\langle \phi \otimes \psi_h, \sum_{1\le i<j\le k} \frac{U}{|x_i - x_j|}(\phi \otimes \psi_h)\right\rangle + \left\langle \phi \otimes \psi_h, \sum_{k+1\le i<j\le n} \frac{U}{|x_i - x_j|}(\phi \otimes \psi_h)\right\rangle + \left\langle \phi \otimes \psi_h, \sum_{\substack{1\le i\le k \\ k+1\le j\le n}} \frac{U}{|x_i - x_j|}(\phi \otimes \psi_h)\right\rangle = \left\langle \phi, \sum_{1\le i<j\le k} \frac{U}{|x_i - x_j|}\phi\right\rangle + \left\langle \psi_h, \sum_{k+1\le i<j\le n} \frac{U}{|x_i - x_j|}\psi_h\right\rangle + \tag{7}$$

$$\left\langle \phi \otimes \psi_h, \sum_{\substack{1\le i\le k \\ k+1\le j\le n}} \frac{U}{|x_i - x_j|}(\phi \otimes \psi_h)\right\rangle \longrightarrow \left\langle \phi, \sum_{1\le i<j\le k} \frac{U}{|x_i - x_j|}\phi\right\rangle + \left\langle \psi, \sum_{k+1\le i<j\le n} \frac{U}{|x_i - x_j|}\psi\right\rangle + 0 \tag{8}$$

as $|h| \to \infty$. This is true since we may estimate all the terms $\frac{1}{|x_i - x_j|}$ in (8) by the expression

$$\frac{1}{\text{dist}(\text{supp}(\phi), \text{supp}(\psi_h))}$$

which goes to 0 if we let $|h| \to \infty$.

For the third term of the RHS of (5) we first have to notice that

$$\rho_{\phi\otimes\psi_h} = \rho_\phi + \rho_{\psi_h} \tag{9}$$

which implies

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_{\phi\otimes\psi_h}(x)\rho_{\phi\otimes\psi_h}(y)}{|x-y|}\, dxdy = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(\rho_\phi(x) + \rho_{\psi_h}(x))(\rho_\phi(y) + \rho_{\psi_h}(y))}{|x-y|}\, dxdy = \tag{10}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_\phi(x)\rho_\phi(y)}{|x-y|}\ dxdy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_{\psi_h}(x)\rho_{\psi_h}(y)}{|x-y|}\ dxdy + \tag{11}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_\phi(x)\rho_{\psi_h}(y)}{|x-y|}\ dxdy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_{\psi_h}(x)\rho_\phi(y)}{|x-y|}\ dxdy \longrightarrow \tag{12}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_\phi(x)\rho_\phi(y)}{|x-y|}\ dxdy + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_\psi(x)\rho_\psi(y)}{|x-y|}\ dxdy + 0 + 0$$

as $|h| \to \infty$. This is true since we have that $\operatorname{dist}(\operatorname{supp}(\phi), \operatorname{supp}(\psi_h)) \to 0$ as $|h| \to \infty$.

Putting everything together implies (4). From (4) we have

$$\mathcal{E}_n \leq \lim_{|h|\to\infty} P_n(\phi \otimes \psi_h) = P_k(\phi) + P_{n-k}(\psi) < \mathcal{E}_k + \mathcal{E}_{n-k} + \epsilon$$

and since this holds for all $\epsilon > 0$ the desired estimate follows.

□