Exercise 36. By definition and Newton's theorem we have

$$\Big\langle \phi\otimes\phi, \frac{1}{|z_1-z_2-y|}\ \phi\otimes\phi\Big\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \overline{\phi(z_1)\phi(z_2)}\ \frac{\phi(z_1)\phi(z_2)}{|z_1-z_2-y|}\ dz_1dz_2 =$$

$$\int_{\mathbb{R}^3} |\phi(z_2)|^2 \int_{\mathbb{R}^3} \frac{1}{|z_1-z_2-y|}\ |\phi(z_1)|^2\ dz_1dz_2 =$$

$$\int_{\mathbb{R}^3} |\phi(z_2)|^2 \int_{\mathbb{R}^3} \frac{1}{\max\{|z_1|,|z_2+y|\}}\ |\phi(z_1)|^2\ dz_1dz_2 \tag{1}$$

and since the function ϕ is supported in the ball $B(0,\frac{|y|}{3})$ we have that $z_1, z_2 \in B(0,\frac{|y|}{3})$ which implies

$$|z_1| \le \frac{|y|}{3} \le |y| - |z_2| \le ||y|-|z_2|| \le |z_2+y|$$

by the inverse triangle inequality. Hence, (1) is equal to

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\phi(z_1)|^2|\phi(z_2)|^2}{|z_2+y|}\ dz_1dz_2 = \Big\langle \phi\otimes\phi, \frac{1}{|z_2+y|}\ \phi\otimes\phi\Big\rangle$$

and the proof is complete.

□

Exercise 37. Consider a standard bump function $\Phi \in C_c^\infty(\mathbb{R}^3)$ such that $0 \le \Phi(x) \le 1$ for all $x \in \mathbb{R}^3$, $\Phi(x) = 1$ for $|x| \le \frac{1}{3}$ and $\Phi(x) = 0$ for $|x| \ge \frac{1}{2}$. Define

$$\Phi_y(x) = \Phi\Big(x - \frac{y}{|y|}\Big) \text{ and } \Psi = \sqrt{2-\Phi^2}.$$

Notice that since $0 \le \Phi \le 1$ the quantity $2 - \Phi^2$ is always positive which implies that the function $\Psi \in C^\infty(\mathbb{R}^3)$. Next we define

$$\phi = \Phi^2 \text{ so that } \Psi^2 = 2 - \phi$$

and then we let

$$j_1(x) = \Phi(x)\Psi(x) \text{ and } j_2(x) = \Phi_y(x)\Psi_y(x)$$

where $\Psi_y(x) = \Psi(x - \frac{y}{|y|})$. Finally, let

$$j_3(x) = 1 - \phi(x) - \phi_y(x)$$

where $\phi_y(x) = \phi(x - \frac{y}{|y|})$.

Observe that $j_1, j_2 \in C_c^\infty(\mathbb{R}^3)$, $j_3 \in C^\infty(\mathbb{R}^3)$ with values in $[0,1]$ and

$$j_3^2(x) = (1-\phi(x)-\phi_y(x))^2 = 1 + \phi(x)^2 + \phi_y(x)^2 - 2\phi(x) - 2\phi_y(x) + 2\phi(x)\phi_y(x) =$$

$$1 + \phi(x)(\phi(x)-2) + \phi_y(x)(\phi_y(x)-2) = 1 - j_1^2(x) - j_2^2(x)$$

where we used the fact that since the functions ϕ, ϕ_y have disjoint supports the function $\phi\phi_y \equiv 0$. The proof is complete.

□

Exercise 38. Let us denote by

$$H_0 = \frac{1}{|y|},\ H_1 = -\Delta_{x_1} - \frac{1}{|x_1|} - \frac{1}{|x_1 - y|} + \frac{1}{|y|} \tag{2}$$

and

$$H_2 = H_1 - \Delta_{x_2} - \frac{1}{|x_2|} - \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|}. \tag{3}$$

Our goal is to show the HVZ theorem

$$\inf \sigma_{\text{ess}}(H_1) = \inf \sigma(H_0) \text{ and } \inf \sigma_{\text{ess}}(H_2) = \inf \sigma(H_1) \tag{4}$$

and then Zishlin's theorem, i.e.

$$\inf \sigma(H_1) < \inf \sigma(H_0) \text{ and } \inf \sigma(H_2) < \inf \sigma(H_1). \tag{5}$$

From (4) and (5) the required result follows since we have

$$\inf \sigma(H_2) < \inf \sigma(H_1) = \inf \sigma_{\text{ess}}(H_2).$$

Hence, our goal from now on is to prove (4) and (5).

We start with the proof of (4). We will only show the second part, that is the equality $\inf \sigma_{\text{ess}}(H_2) = \inf \sigma(H_1)$ since the argument for the first part of (4) is similar and actually simpler. Let $E_1 = \inf \sigma(H_1)$. As in the proof of the HVZ theorem from 1st semester it suffices to show

$$[E_1, \infty) \subset \sigma(H_2) \tag{6}$$

and

$$E_1 \leq \inf \sigma_{\text{ess}}(H_2). \tag{7}$$

Consider $\epsilon > 0$ and $\lambda = E_1 + \delta$ for some $\delta \geq 0$. We can find $\psi \in C_c^\infty(\mathbb{R}^3)$ such that $\|\psi\|_2 = 1$ and

$$\|(H_1 - E_1)\psi\| < \frac{\epsilon}{3} \tag{8}$$

and let $\phi \in C_c^\infty(\mathbb{R}^3)$ with $\|\phi\|_2 = 1$ and

$$\|(-\Delta_{x_2} - \delta)\phi\| < \frac{\epsilon}{3}. \tag{9}$$

For $\phi_h = \phi(\cdot - h)$ we have $\|\psi \otimes \phi_h\| = 1$. Then

$$\begin{aligned} \|(H_2 - \lambda)(\psi \otimes \phi_h)\| &\leq \|(H_1 - E_1)(\psi \otimes \phi_h)\| + \|(\Delta_{x_2} - \delta)(\psi \otimes \phi_h)\| + \\ &\left\| \left(-\frac{1}{|x_2|} - \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|} \right)(\psi \otimes \phi_h) \right\| \leq \\ &\frac{2\epsilon}{3} + \left\| \left(-\frac{1}{|x_2|} - \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|} \right)(\psi \otimes \phi_h) \right\| < \epsilon \end{aligned} \tag{10}$$

for $|h|$ large by estimating the denominators of the last expression by the quantities $\mathrm{dist}(\mathrm{supp}\phi_h, 0)$ and $\mathrm{dist}(\mathrm{supp}\phi_h, \mathrm{supp}\psi)$. Thus, (6) is proved. For (7) we use the IMS localization formula to write

$$H_2 = \sum_{k=0}^{2} J_{k,R} H_2 J_{k,R} - \sum_{k=0}^{2} |\nabla J_{k,R}|^2 \tag{11}$$

where J_1, J_2, J_3 are bump functions as in the proof of the HVZ theorem from last semester and $J_{k,R}(x) = J_k(\frac{x}{R})$ and $R > 0$. Then we just follow the proof given last semester which uses Weyl's criterion to close the argument. So in this way we have the proof of (7) and consequently the proof of (4).

We continue with the proof of (5). The first inequality in (5) follows from the simple observation that

$$\inf \sigma\Big(\Delta_{x_1} - \frac{1}{|x_1|} - \frac{1}{|x_1 - y|}\Big) \leq \inf \sigma\Big(- \Delta_{x_1} - \frac{1}{|x_1|}\Big) = -\frac{1}{4} < 0 \tag{12}$$

where for the last equality we used Theorem 5.5 from last semester. Notice that this implies the existence of a ground state ϕ, $\|\phi\| = 1$, of H_1 since

$$\inf \sigma(H_1) < \inf \sigma(H_0) = \inf \sigma_{\text{ess}}(H_1).$$

For the second inequality in (5) we follow the proof of Zishlin's theorem from last semester. Indeed, for a ψ we define $\psi_n(x) = 2^{-\frac{3n}{2}} \psi(2^{-n}x)$ and it suffices to show that for infinitely many $n \in \mathbb{N}$ we have

$$\Big\langle \phi \otimes \psi_n, H_2(\phi \otimes \psi_n)\Big\rangle < \inf \sigma(H_1). \tag{13}$$

Since

$$H_2 = H_1 - \Delta_{x_2} - \frac{1}{|x_2|} - \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|}$$

we have

$$\Big\langle \phi \otimes \psi_n, H_2(\phi \otimes \psi_n)\Big\rangle = \Big\langle \phi \otimes \psi_n, H_1(\phi \otimes \psi_n)\Big\rangle + \Big\langle \phi \otimes \psi_n, \Big(- \Delta_{x_2} - \frac{1}{|x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle + \tag{14}$$

$$\Big\langle \phi \otimes \psi_n, \Big(- \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle =$$

$$\Big\langle \phi, H_1\phi\Big\rangle + \Big\langle \phi \otimes \psi_n, \Big(- \Delta_{x_2} - \frac{1}{|x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle +$$

$$\Big\langle \phi \otimes \psi_n, \Big(- \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle =$$

$$\inf \sigma(H_1) + \Big\langle \phi \otimes \psi_n, \Big(- \Delta_{x_2} - \frac{1}{|x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle + \tag{15}$$

$$\Big\langle \phi \otimes \psi_n, \Big(- \frac{1}{|x_2 - y|} + \frac{1}{|x_1 - x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle.$$

At this point we choose the function ψ to be spherically symmetric (which implies that the function ψ_n is also spherically symmetric) and then apply Newton's theorem to arrive at the upper bound

$$\inf \sigma(H_1) + \Big\langle \phi \otimes \psi_n, \Big(-\Delta_{x_2} - \frac{1}{|x_2|}\Big)(\phi \otimes \psi_n)\Big\rangle - \Big\langle \phi \otimes \psi_n, \frac{1}{|x_2 - y|}\phi \otimes \psi_n \Big\rangle + \Big\langle \phi \otimes \psi_n, \frac{1}{|x_2|}\phi \otimes \psi_n \Big\rangle = \tag{16}$$

$$\inf \sigma(H_1) + \Big\langle \phi \otimes \psi_n, (-\Delta_{x_2})(\phi \otimes \psi_n)\Big\rangle - \Big\langle \phi \otimes \psi_n, \frac{1}{|x_2 - y|}\phi \otimes \psi_n \Big\rangle. \tag{17}$$

From the mean value theorem we may write

$$\frac{1}{|x_2 - y|} = \frac{1}{|x_2|} - \xi \ \frac{y \cdot \hat{x}_2}{|x_2|^2}$$

for some $\xi \in (0,1)$ where $\hat{x}_2 = \frac{x_2}{|x_2|}$. Hence, we continue from (17) as

$$\inf \sigma(H_1) + \Big\langle \phi \otimes \psi_n, (-\Delta_{x_2})(\phi \otimes \psi_n)\Big\rangle - \Big\langle \phi \otimes \psi_n, \frac{1}{|x_2|}\phi \otimes \psi_n \Big\rangle + \Big\langle \phi \otimes \psi_n, \xi \ \frac{y \cdot \hat{x}_2}{|x_2|^2}\phi \otimes \psi_n \Big\rangle = \tag{18}$$

$$\inf \sigma(H_1) + \frac{1}{2^n}\Big(\frac{1}{2^n}\Big\langle \psi, \Delta\psi \Big\rangle - \Big\langle \phi \otimes \psi, \frac{1}{|x_2|}\phi \otimes \psi \Big\rangle + \frac{1}{2^n}\Big\langle \phi \otimes \psi, \xi \ \frac{y \cdot \hat{x}_2}{|x_2|^2}\phi \otimes \psi \Big\rangle\Big)$$

which for large $n \in \mathbb{N}$ this last expression is strictly bounded from above by $\inf \sigma(H_1)$ which finishes the proof.

□