Exercise 39. We use the Taylor expansion around 0 of the function

$$f(z) = \frac{1}{|z+y|}$$

where $|z| \leq \frac{2|y|}{3}$ i.e. we write

$$f(z) = f(0) + \Big\langle z, \nabla f(0) \Big\rangle + \frac{1}{2}\Big\langle z, Hf(0)z \Big\rangle + \text{higher order terms.}$$

It is straightforward to see that ($z_1, z_2, z_3 \in \mathbb{R}$)

$$\nabla f(z_1, z_2, z_3) = -\Big(\frac{z_1 + y_1}{|z+y|^3}, \frac{z_2 + y_2}{|z+y|^3}, \frac{z_3 + y_3}{|z+y|^3}\Big)$$

where $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. Similarly, we find the Hessian matrix

$$Hf(z) = \Big(\partial_i \partial_j f(z)\Big)_{1 \leq i,j \leq 3}$$

of f and evaluate all of these quantities at 0. The required expression follows.

$\square$

Exercise 40. By writing $H = \tilde{H}_0 + I$ we have that the LHS is equal to

$$\Big\langle P^\perp(\tilde{H}_0 + I)\phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp (\tilde{H}_0 + I)\phi \otimes \phi_y \Big\rangle =$$

$$\Big\langle P^\perp \tilde{H}_0 \phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp \tilde{H}_0 \phi \otimes \phi_y \Big\rangle + \Big\langle P^\perp \tilde{H}_0 \phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp I \phi \otimes \phi_y \Big\rangle +$$

$$\Big\langle P^\perp I \phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp \tilde{H}_0 \phi \otimes \phi_y \Big\rangle + \Big\langle P^\perp I \phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp I \phi \otimes \phi_y \Big\rangle.$$

At this point we use that $\phi \otimes \phi_y$ is almost an eigenfunction of the operator $\tilde{H}_0$ i.e.

$$P^\perp \tilde{H}_0 \phi \otimes \phi_y = P^\perp E(\infty)\phi \otimes \phi_y + O(e^{-c|y|}) = E(\infty) P^\perp \phi \otimes \phi_y + O(e^{-c|y|}) = O(e^{-c|y|})$$

the fact that

$$\|(\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp I \phi \otimes \phi_y\| \leq \frac{c}{|y|^3}$$

and we arrive at the expression

$$\Big\langle P^\perp I \phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp I \phi \otimes \phi_y \Big\rangle + O(e^{-c|y|}).$$

After a change of variables ($z + y \to \tilde{z}$) and an application of Newton's theorem (as it was done in class) we obtain that

$$\Big\langle P^\perp I \phi \otimes \phi_y, (\tilde{H}_0^\perp - E(\infty))^{-1} P^\perp I \phi \otimes \phi_y \Big\rangle = \Big\langle I \phi \otimes \phi, (H_0^\perp - E(\infty))^{-1} I \phi \otimes \phi \Big\rangle$$

which is the required expression.

$\square$

Exercise 41. We start by observing that

$$e^{\delta}(-\Delta+1)^{-1}e^{-\delta} = (e^{\delta}(-\Delta+1)e^{-\delta})^{-1} = (e^{\delta}(-\Delta)e^{-\delta}+1)^{-1}.$$

Then for a function f we have

$$e^{\delta}(-\Delta)e^{-\delta}f = -\Delta f + 2e^{\delta}e^{-\delta}\nabla\delta\nabla f - e^{\delta}(\Delta e^{-\delta})f =$$
$$-\Delta f + 2\nabla\delta\nabla f - e^{\delta}(e^{-\delta}|\nabla\delta|^2 + e^{-\delta} + \Delta\delta)f = (-\Delta + 2\nabla\delta\nabla - |\nabla\delta|^2 - \Delta\delta)f$$

which means we have to prove the boundedness of the operator

$$(-\Delta + 2\nabla\delta\nabla - |\nabla\delta|^2 - \Delta\delta + 1)^{-1}.$$

But by calculating $\nabla\delta, \Delta\delta$ we may write

$$-\Delta + 2\nabla\delta\nabla - |\nabla\delta|^2 - \Delta\delta + 1 = -\Delta + c\Phi\nabla + c\Psi + 1 \tag{1}$$

where Φ and Ψ are bounded functions and c is the constant appearing in the definition of δ. Observe that the RHS of (1) is equal to

$$-\Delta + 1 + c(\Phi\nabla + c\Psi) = [\mathrm{Id} + c(\Phi\nabla + c\Psi)(-\Delta+1)^{-1}](-\Delta+1). \tag{2}$$

Hence, if we show that the operator $\nabla(-\Delta+1)^{-1}$ is bounded (all other operators inside the brackets[ ] are bounded) then for sufficiently small $c > 0$ the RHS of (2) is an invertible bounded operator with bounded inverse and the proof will be complete.

At this point we recall that from the first part of the class (see end of lecture 7 and beginning of lecture 8) we know that for any $\epsilon > 0$ we can find a constant $c_\epsilon > 0$ such that

$$\|\nabla\psi\| \le \epsilon\| - \Delta\psi\| + c_\epsilon\|\psi\|. \tag{3}$$

By (3) we have

$$\|\nabla(-\Delta+1)^{-1}\phi\| \le \epsilon\|(-\Delta)(-\Delta+1)^{-1}\phi\| + c_\epsilon\|(-\Delta+1)^{-1}\phi\|$$

which implies that the operator $\nabla(-\Delta+1)^{-1}$ is bounded and we are done.

$\square$